

11. How many words can be formed using all letters (using each letter exactly once) of the following words:

(i) DELHI

(ii) JAIPUR

(iii) HEXAGON

(iv) EQUATION

Answers

1. (i) 24	(ii) 4160	(iii) 2520	(iv) 720		
3. (i) 5	(ii) 3	(iii) 8	4. (i) 30	(ii) 120	
5. (i) 504	(ii) 729	6. (i) 720	(ii) 5040	7. (i) 56	(ii) 11880
8. (i) 63	(ii) 256	9. 24	10. (i) 3	(ii) 9	(iii) 5
11. (i) 120	(ii) 720	(iii) 5040	(iv) 40320		

PERMUTATIONS WHEN ALL OBJECTS ARE NOT DIFFERENT

Theorem. The number of permutations of n objects taken all together, when p of the objects are alike of one kind, q of them alike of another kind, r of them alike of a third kind and the remaining all different is $\frac{n!}{p!q!r!}$.

Proof. Let the required number of permutations be x . Now, if we replace p alike objects with p distinct objects, without altering the position of other objects, we get $x \times p!$ arrangements as p distinct objects can be arranged in $p!$ ways within themselves. Now, if q alike objects are also replaced by q distinct objects, without altering the position of other objects, we get $x \times p!q!$ arrangements. Finally, if r alike objects are also replaced by r distinct objects, we get $x \times p!q!r!$ arrangements. Now all the n objects are distinct and hence number of arrangements should be $n!$.

$$\text{Hence, } x \times p!q!r! = n! \Rightarrow x = \frac{n!}{p!q!r!}.$$

Corollary. The number of permutations of n objects, of which m are of one kind and the rest $n - m$ of another kind, taken all at a time is $\frac{n!}{m!(n-m)!}$.

ILLUSTRATIVE EXAMPLES

Example 1. In how many ways can 4 red, 3 yellow and 2 green discs can be arranged in a row if discs of the same colour are indistinguishable?

Solution. In total, we have $4 + 3 + 2$ i.e. 9 discs of which 4 are of one kind (red), 3 of another kind (yellow) and 2 of another kind (green).

The number of ways of arranging these discs in a row

$$= \frac{9!}{4!3!2!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 6 \cdot 2} = 1260.$$

Example 2. Find the number of permutations of the letters of the word HYDERABAD.

Solution. Here, the given word has 9 letters of which there are two D's, two A's and the remaining 5 are all different.

$$\begin{aligned} \text{Therefore, the required number of permutations} &= \frac{9!}{2!2!} \\ &= \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 90720. \end{aligned}$$

Example 3. How many numbers greater than 1000000 can be formed by using the digits 1, 2, 0, 2, 4, 2, 4?

Solution. There are seven digits—one 1, three 2's, one 0 and two 4's. Since the number 1000000 has seven digits and the number greater than 1000000 must have atleast 7 digits, so the required numbers are 7-digit numbers only. All seven digit numbers formed from given digits are greater than 1000000.

$$\text{The number of arrangements of given digits} = \frac{7!}{3! \times 2!}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 420.$$

But these include numbers which have 0 on the extreme left and hence are not of seven digits.

$$\text{The number of such numbers} = \frac{6!}{3! \times 2!} = \frac{720}{6 \times 2} = 60.$$

Hence, the required number of numbers = $420 - 60 = 360$.



EXERCISE 7.4

- In how many ways can 3 white, 4 red and 1 blue marbles be arranged in a row if marbles of the same colour are indistinguishable?
- How many different signals can be transmitted by hoisting 3 red, 4 yellow and 2 blue flags on a pole, assuming that in transmitting a signal all nine flags are to be used and flags of the same colour are indistinguishable?
- Find the number of arrangements that can be made out of the following words:

(i) ROOT	(ii) BANANA	(iii) APPLE	(iv) PINEAPPLE
(v) INSTITUTE	(vi) CIVILIZATION.	(vii) COMMERCE	

Answers

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- | | | | | | | |
|-----------|---------|----------|------------|-----------|---------------|------------|
| 1. 280 | 2. 1260 | | | | | |
| 3. (i) 12 | (ii) 60 | (iii) 60 | (iv) 30240 | (v) 30240 | (vi) 19958400 | (vii) 5040 |
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COMBINATIONS

Each of the groups or selections which can be made by taking some or all of a number of objects without reference to the order of objects in each group is called a combination.

For example:

- (i) The combinations *i.e.* selections which can be made from the letters a, b, c by taking 2 at a time are ab, bc, ca .

Note that ab and ba represent the same selection because each of these selections contain the same letters a and b .

Thus, the number of combinations of 3 different objects taken 2 at a time is 3.

- (ii) The only combination *i.e.* selection which can be made from the letters a, b, c taken all at a time is abc .

Thus, the number of combinations of 3 different objects taken all at a time is 1.

- (iii) The combinations *i.e.* groups which can be made from the letters a, b, c, d by taking 3 at a time are abc, abd, acd, bcd .

Note that abc, acb, bac, bca, cab and cba represent the same group because each of these groups contain the same letters a, b and c .

Thus, the number of combinations of 4 different objects taken 3 at a time is 4.

The difference between a permutation and a combination of objects is that 'order' does matter in a permutation, while it does not matter in case of a combination.

Let r and n be two positive integers and $1 \leq r \leq n$, then the number of combinations of n different objects taken r at a time is denoted by nC_r or $C(n, r)$.

Combinations when all objects are different

Theorem. The number of combinations of n different objects taken r at a time is ${}^nC_r = \frac{|n|}{|r| |n-r|}$.

Proof. Number of combinations of n different objects taken r at a time is nC_r . Take any one of these combinations. It contains r objects which can be arranged among themselves in $|r|$ ways. Thus, one combination gives rise to $|r|$ permutations.

$\therefore {}^nC_r$ combinations will give rise to ${}^nC_r \times |r|$ permutations.

But we know that the number of permutations of n different objects taken r at a time is nP_r .

$$\therefore {}^nC_r \times |r| = {}^nP_r \Rightarrow {}^nC_r = \frac{{}^nP_r}{|r|} = \frac{|n|}{|r| |n-r|}.$$

Corollary 1. (i) ${}^nC_r = \frac{n(n-1)(n-2)\dots \text{upto } r \text{ factors}}{|r|}$ (ii) ${}^nC_n = 1$ (iii) ${}^nC_r = {}^nC_{n-r}$.

Proof. (i) ${}^nC_r = \frac{|n|}{|r| |n-r|} = \frac{n(n-1)(n-2)\dots(n-r+1)|n-r|}{|r| |n-r|}$
 $= \frac{n(n-1)(n-2)\dots \text{upto } r \text{ factors}}{|r|}.$

(ii) ${}^nC_n = \frac{|n|}{|n-n| |n|} = \frac{|n|}{|0| |n|} = 1.$

(iii) ${}^nC_{n-r} = \frac{|n|}{|n-r| |n-(n-r)|} = \frac{|n|}{|n-r| |r|} = {}^nC_r.$

The last result is of great use in computing nC_r when r is large. For example, we can see

that ${}^{11}C_9 = {}^{11}C_2 = \frac{11 \times 10}{1 \times 2} = 55.$

Corollary 2. ${}^nC_a = {}^nC_b \Rightarrow a = b$ or $a + b = n$.

Proof. ${}^nC_a = {}^nC_b \Rightarrow {}^nC_a = {}^nC_b = {}^nC_{n-b}$ ($\because {}^nC_r = {}^nC_{n-r}$)

$\Rightarrow a = b$ or $a = n - b$

$\Rightarrow a = b$ or $n = a + b.$

Corollary 3. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r.$

Proof. ${}^nC_r + {}^nC_{r-1} = \frac{|n|}{|n-r| |r|} + \frac{|n|}{|n-(r-1)| |r-1|} = \frac{|n|}{|n-r| |r|} + \frac{|n|}{|n-r+1| |r-1|}$
 $= \frac{|n|}{|n-r| |r-1|} \left[\frac{1}{r} + \frac{1}{n-r+1} \right]$
 $= \frac{|n|}{|n-r| |r-1|} \cdot \frac{n-r+1+r}{r(n-r+1)} = \frac{(n+1)|n|}{r |r-1| (n-r+1) |n-r|}$
 $= \frac{|n+1|}{|r| |n+1-r|} = {}^{n+1}C_r.$

Remark

We define ${}^nC_0 = 1$ i.e. the number of combinations of n different objects taken nothing at all is taken as 1.

Counting the number of combinations is merely counting the number of ways in which some or all objects at a time are selected. Selecting no objects at all is the same thing as leaving behind all the objects and we know that there is only one way of doing so. That is why we have defined ${}^nC_0 = 1$.

Thus, the formula ${}^nC_r = \frac{|n|}{|r| |n-r|}$ is meaningful for $r = 0$ also.

ILLUSTRATIVE EXAMPLES

Example 1. Evaluate the following:

(i) ${}^{12}C_5$ (ii) ${}^{10}C_8$ (iii) ${}^{10}C_7 + {}^{10}C_6$ (iv) ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$.

Solution. (i) ${}^{12}C_5 = \frac{|12|}{|5| |7|} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1 \times 7} = 792$.

(ii) ${}^{10}C_8 = {}^{10}C_2$ (using ${}^nC_r = {}^nC_{n-r}$)
 $= \frac{10 \times 9}{1 \times 2} = 45$.

(iii) ${}^{10}C_7 + {}^{10}C_6 = {}^{11}C_7$ (using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$)
 $= {}^{11}C_4$ (using ${}^nC_r = {}^nC_{n-r}$)
 $= \frac{11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4} = 330$.

(iv) ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7 = ({}^{15}C_8 + {}^{15}C_9) - ({}^{15}C_6 + {}^{15}C_7) = {}^{16}C_9 - {}^{16}C_7 = 0$ ($\because {}^{16}C_9 = {}^{16}C_7$)

Example 2. Find n if

(i) ${}^nC_5 = {}^nC_7$ (ii) $16 \cdot {}^{n+2}C_8 = 57 \cdot {}^{n-2}P_4$ (iii) $n \times {}^{10}C_4 = {}^{10}P_4$.

Solution. (i) We can directly use the formula:

$${}^nC_a = {}^nC_b \Rightarrow \text{either } a = b \text{ or } n = a + b.$$

Hence, ${}^nC_5 = {}^nC_7 \Rightarrow n = 5 + 7 = 12$ ($\because 5 \neq 7$)

(ii) $16 \cdot {}^{n+2}C_8 = 57 \cdot {}^{n-2}P_4 \Rightarrow 16 \cdot \frac{|n+2|}{|n+2-8| |8|} = 57 \cdot \frac{|n-2|}{|n-2-4|}$

$$\Rightarrow \frac{16 \cdot |n+2|}{|8| |n-6|} = 57 \cdot \frac{|n-2|}{|n-6|} \Rightarrow \frac{|n+2|}{|n-2|} = \frac{57 \cdot |8|}{16}$$

$$\Rightarrow (n+2)(n+1)n(n-1) = \frac{57 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{16} = 19 \cdot 3 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 1$$

$$= 21 \cdot 20 \cdot 19 \cdot 18$$

$$\Rightarrow n+2 = 21 \Rightarrow n = 19$$

(iii) $n \times {}^{10}C_4 = {}^{10}P_4 \Rightarrow n \times {}^{10}C_4 = {}^{10}C_4 \times |4|$

$$\Rightarrow n = |4| \Rightarrow n = 24.$$

($\because {}^nP_r = {}^nC_r \times |r|$)

Example 3. (i) Find r if ${}^{11}C_{2r} = {}^{11}C_{r+5}$ (ii) Find r if ${}^9C_r - {}^8C_3 = {}^8C_2$

(iii) If ${}^{2n}C_3 : {}^nC_3 = 11 : 1$, find n .

Solution. (i) ${}^{11}C_{2r} = {}^{11}C_{r+5} \Rightarrow 2r = r + 5$ or $2r + r + 5 = 11$

$$\Rightarrow r = 5 \text{ or } r = 2.$$

(ii) Given ${}^9C_r - {}^8C_3 = {}^8C_2 \Rightarrow {}^9C_r = {}^8C_3 + {}^8C_2$

$$\Rightarrow {}^9C_r = {}^9C_3$$

$$\Rightarrow r = 3 \text{ or } 9 = r + 3$$

$$\Rightarrow r = 3 \text{ or } r = 6.$$

$$(\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r)$$

(iii) Given ${}^{2n}C_3 : {}^nC_3 = 11 : 1 \Rightarrow \frac{|2n}{|3|2n-3} \times \frac{|3||n-3|}{|n|} = \frac{11}{1}$

$$\Rightarrow \frac{2n(2n-1)(2n-2)|2n-3|}{|2n-3|} \times \frac{|n-3|}{n(n-1)(n-2)|n-3|} = \frac{11}{1}$$

$$\Rightarrow \frac{2n(2n-1)(2n-2)}{n(n-1)(n-2)} = \frac{11}{1}$$

$$\Rightarrow \frac{4(2n-1)(n-1)}{(n-1)(n-2)} = \frac{11}{1} \Rightarrow \frac{4(2n-1)}{n-2} = \frac{11}{1}$$

$$\Rightarrow 11n - 22 = 8n - 4 \Rightarrow 3n = 18$$

$$\Rightarrow n = 6.$$

Example 4. If nC_4 , nC_5 and nC_6 are in A.P., find n .

Solution. Given nC_4 , nC_5 , nC_6 are in A.P.

$$\Rightarrow 2 \cdot {}^nC_5 = {}^nC_4 + {}^nC_6$$

$$\Rightarrow 2 \cdot \frac{|n}{|5||n-5|} = \frac{|n}{|4||n-4|} + \frac{|n}{|6||n-6|}$$

$$\Rightarrow \frac{2 \cdot |n|}{5 \cdot |4| \cdot (n-5)|n-6|} = \frac{|n|}{|4| \cdot (n-4)(n-5)|n-6|} + \frac{|n|}{6 \cdot 5 \cdot |4| \cdot |n-6|}$$

$$\Rightarrow \frac{2}{5(n-5)} = \frac{1}{(n-4)(n-5)} + \frac{1}{30}$$

$$\Rightarrow \frac{2}{5(n-5)} - \frac{1}{(n-4)(n-5)} = \frac{1}{30} \Rightarrow \frac{2(n-4)-5}{5(n-4)(n-5)} = \frac{1}{30}$$

$$\Rightarrow \frac{2n-13}{n^2-9n+20} = \frac{1}{6} \Rightarrow n^2-9n+20 = 12n-78$$

$$\Rightarrow n^2-21n+98=0 \Rightarrow (n-7)(n-14)=0$$

$$\Rightarrow n = 7 \text{ or } 14$$

Example 5. Prove that $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$.

$$\begin{aligned} \text{Solution. L.H.S.} &= \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{\frac{|n}{|r||n-r|}}{\frac{|n}{|r-1||n-r+1|}} = \frac{|n|}{|r||n-r|} \times \frac{|r-1||n-r+1|}{|n|} \\ &= \frac{|r-1| \cdot (n-r+1) |n-r|}{r |r-1| |n-r|} = \frac{n-r+1}{r} = \text{R.H.S.} \end{aligned}$$

Example 6. From a class of 36 students, 3 students are to be selected to take part in a competition. How many such selections can be made?

Solution. The required number of selections = ${}^{36}C_3$

$$= \frac{36 \times 35 \times 34}{1 \times 2 \times 3} = 7140.$$

Example 7. A polygon has 44 diagonals. Find the number of its sides.

Solution. Let the polygon have n sides, then

number of diagonals = total number of straight lines – number of sides

$$\Rightarrow 44 = {}^nC_2 - n \Rightarrow 44 = \frac{n(n-1)}{1 \cdot 2} - n \Rightarrow 44 = n \left(\frac{n-1}{2} - 1 \right)$$

$$\Rightarrow 44 = \frac{n(n-3)}{2} \Rightarrow n^2 - 3n - 88 = 0 \Rightarrow (n-11)(n+8) = 0$$

$$\Rightarrow n = 11 \text{ or } -8 \text{ but } n \text{ cannot be negative}$$

$$\Rightarrow n = 11.$$

Hence, the polygon has 11 sides.

Example 8. If m parallel lines in a plane are intersected by a family of n parallel lines, find the number of parallelograms formed.

Solution. A parallelogram is formed by choosing any two lines from set of m parallel lines and then any two lines from the set of n parallel lines. Hence, number of parallelograms formed

$$= {}^mC_2 \times {}^nC_2 = \frac{m(m-1)n(n-1)}{4}.$$

Example 9. Everybody in a room shakes hands with everybody else. The total number of shake hands is 66. How many people are there in the room?

Solution. Let there be n people in the room.

When two persons shake hands, it gives rise to one shake hand.

So when n persons shake hands, the number of shake hands = nC_2 .

According to given, ${}^nC_2 = 66$

$$\Rightarrow \frac{n(n-1)}{1 \times 2} = 66$$

$$\Rightarrow n(n-1) = 132 \Rightarrow n(n-1) = 12 \times 11 \Rightarrow n = 12.$$



EXERCISE 7.5

1. Evaluate the following:

(i) ${}^{15}C_{11}$

(ii) ${}^{100}C_{98}$

(iii) ${}^{52}C_{52}$.

2. Evaluate the following:

(i) ${}^2C_0 + {}^3C_1 + {}^4C_2$

(ii) ${}^{19}C_{17} + {}^{19}C_{18}$.

3. If ${}^{35}C_{n+7} = {}^{35}C_{4n-2}$, find the value(s) of n .

4. (i) If ${}^nC_{10} = {}^nC_{12}$, find nC_3

(ii) If ${}^{16}C_r = {}^{16}C_{r+2}$, find rC_4 .

5. If ${}^nP_r = 720 \times {}^nC_r$, find the value of r .

6. (i) How many selections of 4 books can be made from 8 different books?

(ii) How many selections of 9 stamps can be made from 11 different stamps?

7. (i) A baseball team has 13 members. How many lineups of 9 players are possible? The position of each member in the lineup is not important.

- (ii) Twelve persons meet in a room and each shakes hands with all others. Find the number of handshakes.
8. (i) In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 courses are compulsory for every student?
 (ii) How many different committees each consisting of 3 girls and 2 boys can be chosen from 7 girls and 5 boys?
9. (i) A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected from the lot, assuming that balls of the same colour are distinguishable.
 (ii) A book shelf contains 7 different mathematics books and 5 different physics books. How many groups of 3 mathematics and 3 physics books can be selected?
10. (i) What is the number of ways of choosing 2 red cards from a pack of 52 cards?
 (ii) What is the number of ways of choosing 3 picture cards from a deck of 52 cards?

Answers

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|-------------|-------------|------------|-----------|----------|
| 1. (i) 1365 | (ii) 4950 | (iii) 1 | 2. (i) 10 | (ii) 190 |
| 3. 3 or 6 | 4. (i) 1540 | (ii) 35 | 5. 6 | |
| 6. (i) 70 | (ii) 55 | 7. (i) 715 | (ii) 66 | |
| 8. (i) 35 | (ii) 350 | 9. (i) 200 | (ii) 350 | |
| 10. (i) 325 | (ii) 220 | | | |

MULTIPLE CHOICE QUESTIONS

Choose the correct answer from the given four options in questions (1 to 11):

- The number of 3 digit odd numbers, when repetition of digits is allowed is
 (a) 450 (b) 360 (c) 400 (d) 420
- The number of 4 digit numbers that can be formed with the digits 2, 3, 4, 7 and using each digit only once is
 (a) 120 (b) 96 (c) 24 (d) 100
- The number of 4 letter words with or without meaning that can be formed out of the letters of the word 'WONDER', if repetition of letters is not allowed is
 (a) 24 (b) 6^4 (c) 4^6 (d) 360
- There are 10 lamps in a hall. Each one of them can be switched on independently. The number of ways in which the hall can be illuminated is
 (a) 2^{10} (b) $2^{10} - 1$ (c) 10^2 (d) $10^2 - 1$
- The number of six digit numbers that can be formed by using the digits 1, 2, 1, 2, 0, 2 is
 (a) 50 (b) 60 (c) 110 (d) 10
- The number of signals that can be made by 4 flags of different colours, taking one or more at a time is
 (a) 48 (b) 52 (c) 64 (d) 56
- In an examination there are four multiple choice questions and each question has 4 choices. Number of ways in which a student can fail to get all answer correct is
 (a) 256 (b) 254 (c) 255 (d) 63
- In how many ways a committee consisting of 3 men and 2 women can be chosen from 7 men and 5 women?
 (a) 45 (b) 350 (c) 4200 (d) 230

9. Total number of words formed by 3 vowels and 3 consonants taken from 5 vowels and 5 consonants is equal to
 (a) 720 (b) 7200 (c) 72000 (d) 72
10. Every body in a room shakes hands with everybody else. The total number of handshakes is 21. The total number of persons in the room is
 (a) 6 (b) 7 (c) 8 (d) 9
11. Find the number of parallelograms in the adjoining figure
 (a) 60 (b) 64
 (c) 76 (d) 88



Answers

1. (a) 2. (c) 3. (d) 4. (b) 5. (a) 6. (c) 7. (c)
 8. (b) 9. (c) 10. (b) 11. (a)

FILL IN THE BLANKS

Fill in the blanks in questions (1 to 16):

- For $n = 5, r = 2$, the value of $\frac{{}^n P_r}{{}^n C_r}$ is
- If $\frac{1}{3} + \frac{1}{4} = \frac{n}{5}$, then $n = \dots\dots\dots$
- If ${}^n C_{12} = {}^n C_8$, then $n = \dots\dots\dots$
- The number of possible outcomes when a coin is tossed 6 times is
- If ${}^n P_r = 840, {}^n C_r = 35$, then $r = \dots\dots\dots$
- If ${}^n P_r = 120, {}^n C_r = 20$, then $n = \dots\dots\dots$
- If ${}^{15} C_{3r} = {}^{15} C_{r+3}$, then $r = \dots\dots\dots$
- If ${}^n C_8 = {}^n C_6$, then ${}^n C_2 = \dots\dots\dots$
- ${}^{11} C_4 + {}^{11} C_5 - {}^{11} C_6 - {}^{11} C_7 = \dots\dots\dots$
- The number of permutations of n different objects taken r at time, when repetition are allowed is
- If ${}^n P_3 = {}^{n-1} P_4$, then $n = \dots\dots\dots$
- If ${}^{10} P_r = 720$, then $r = \dots\dots\dots$
- The number of permutations of the letters of the word 'INDIA' is
- The number of ways in which 4 letters can be posted in 5 letter boxes such that all the letters are not posted in the same letter box is
- The number of diagonals in a polygon of 7 sides is
- ${}^n C_r + {}^n C_{r-1} = \dots\dots\dots$

Answers

1. 10 2. 25 3. 20 4. 64 5. 4 6. 6
 7. 3 8. 91 9. 0 10. n^r 11. 6 12. 3
 13. 60 14. 620 15. 14 16. $n+1 C_r$