9 LIMITS AND CONTINUITY OF A FUNCTION

CHAPTER

9.1. ▶ INTRODUCTION

The *limit* and *continuity* of a function are the fundamental concepts in *Calculus*, which is an important branch of Mathematics. *Limit is a number that a function 'approaches' as the independent variable of the function approaches a given value.* Limits are used to examine the behaviour of a function near a point but not at a point.

Limits are fundamental to define continuity. Continuous functions are those functions which do not have any abrupt changes in values, known as discontinuities. One way to test for the continuity of a function is to see whether the graph of the function can be traced without lifting the pen or pencil from the paper. In the present chapter, we shall study algebra of limits, limits involving logarithmic and exponential functions, right handed and left handed limits, continuity of a function at a point and on an interval.

9.2. ▶ LIMITS

Consider a function $f(x) = x^3$. Clearly as x takes values very near to 0, the value of $f(x)$ also approaches towards 0. Then, we say

$$\lim_{x \to 0} f(x) = 0$$

which is read as *limit of $f(x)$ as x tends to zero* (or *x approaches zero*) *is zero.*

The limit of $f(x)$ as $x \to 0$ is regarded as the value of $f(x)$ at $x = 0$.

Meaning of $x \to a$. Let x be a variable and a be a constant. Now, if x assumes values nearer and nearer to a, then we say that '*x tends to a*' or '*x approaches a*' and write it as $x \to a$.

By $x \to a$, we mean that

(*a*) x is never equal to a *i.e.*, $x \neq a$.

(*b*) x can assume values nearer and nearer to a from left or x can assume values nearer and nearer to a from right.

For example, by $x \to 2$, we mean that

(*a*) $x \neq 2$

(*b*) $x = 1{\cdot}999, 1{\cdot}9999, 1{\cdot}99999, \ldots$ to the extent that x can approach to $2 - 0$.

or $\quad x = 2{\cdot}001, 2{\cdot}0001, 2{\cdot}00001, \ldots$ to the extent that x can approach to $2 + 0$.

Now, consider a function $f(x) = |x|, x \neq 0$. Here, $f(x)$ is not defined at $x = 0$ *i.e.*, $f(0)$ is not defined. Clearly, for values of x very near to 0, the value of $f(x)$ approaches towards 0, which is also apparent from the graph of $y = f(x) = |x|, x \neq 0$. (fig. 9.1)

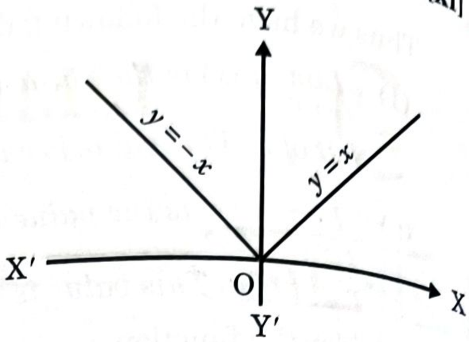

Fig. 9.1

Thus, $\lim_{x \to 0} f(x) = 0$.

Again, consider the function,

$$f(x) = \frac{x^2 - 9}{x - 3}, x \neq 3.$$

Let us find the values of $f(x)$ for x very near to 3, which are given in the following table :

x :	2·9	2·99	2·999	3	3·001	3·01	3·02
$f(x)$:	5·9	5·99	5·999	6	6·001	6·01	6·02

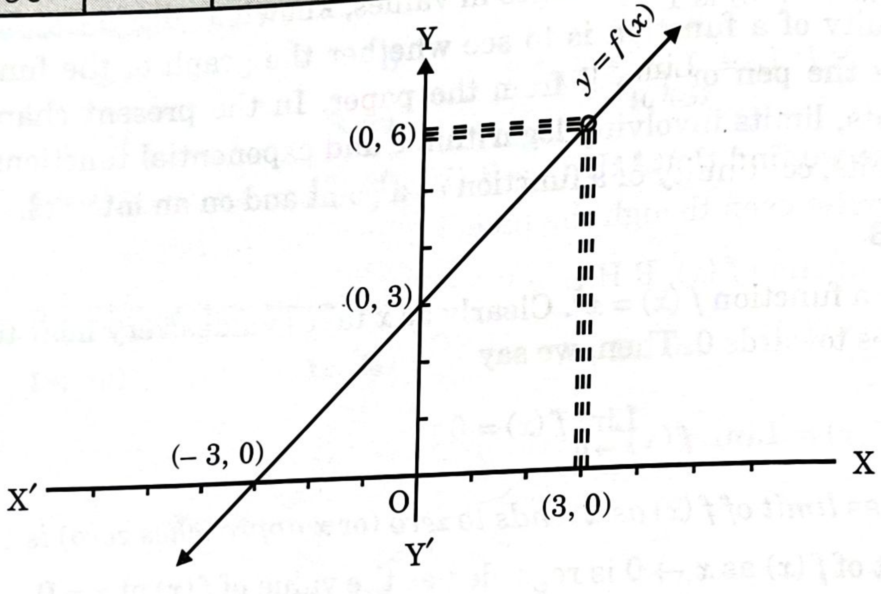

Fig. 9.2

Clearly, all these values are very near to 6 which is also apparent from the graph of $f(x)$ as shown in fig. 9.2

We observe in the above examples that the value of function at any point $x = a$ does not depend on *how* x is approaching towards a. There are infact two ways in which x could approach a number, one from the left and other from the right (*i.e.* for values $< a$ or for values $> a$). Therefore, it leads to two limits *viz.*, the **right hand limit (R.H.L.)** and the **left hand limit (L.H.L.)**.

Right hand limit of a function $f(x)$ is the value of $f(x)$ when x approaches towards a from the right and the left hand limit of a function $f(x)$ is the value of $f(x)$ when x approaches towards a from the left.

Thus we have the following definitions :

(1) $\lim\limits_{x \to a^-} f(x)$ *is the value of* $f(x)$ *at* $x = a$ *obtained by the values of* $f(x)$ *near* x *to the left of* a*. This value is called the left hand limit (L.H.L.) of* $f(x)$ *at* a*.*

(2) $\lim\limits_{x \to a^+} f(x)$ *is the value of* $f(x)$ *at* $x = a$ *obtained by the values of* $f(x)$ *near* x *to the right of* a*. This value is called the right hand limit (R.H.L.) of* $f(x)$ *at* a*.*

Consider the function

$$f(x) = \begin{cases} -1, & x \le 0 \\ 1, & x > 0 \end{cases}$$

The graph of $f(x)$ clearly shows that the value of $f(x)$ when x approaches 0 from the left (*i.e.,* $x \le 0$) is -1

$$\therefore \quad \text{L.H.L.} = \lim_{x \to 0^-} f(x) = -1$$

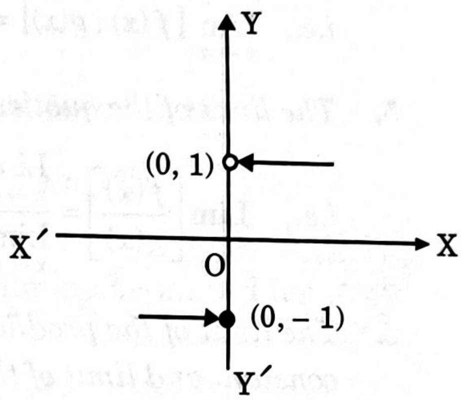

Fig. 9.3

Similarly, the value of $f(x)$ when x approaches 0 from the right (*i.e.,* $x > 0$) is 1

$$\therefore \quad \text{R.H.L.} = \lim_{x \to 0^+} f(x) = 1$$

In this case we find that L.H.L. ≠ R.H.L. so, we say that the limit of $f(x)$ as x tends to zero **does not exist** even though the function is defined at '0'.

If for any function $f(x)$, R.H.L. = L.H.L., then the common value of the limit of $f(x)$ at a is denoted by $\lim\limits_{x \to a} f(x)$

i.e., if $\boldsymbol{\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = l}$**, then** $\boldsymbol{\lim\limits_{x \to a} f(x) = l.}$

Note:

1. $\lim\limits_{x \to a} f(x)$ may exist, even if the function $f(x)$ is not defined at $x = a$.
2. $\lim\limits_{x \to a} f(x)$ may not exist, even if the function is defined at $x = a$.
3. $\lim\limits_{x \to a} f(x)$ may not be equal to $f(a)$ *i.e.,* the limit of a function $f(x)$ as $x \to a$ may be different from the value of the function at $x = a$.

9.3. ▶ ALGEBRA OF LIMITS

We state some theorems (without proof) which will enable us to evaluate the limits easily.

If $f(x)$ and $g(x)$ be two functions such that both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exists, then

1. *The limit of a constant quantity is the constant itself*

   i.e., $\lim\limits_{x \to a} c = c$, where c is a constant.

2. *The limit of the sum of two (or more) functions is equal to the sum of their limits*

i.e., $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$.

3. *The limit of the difference of two functions is equal to the difference of their limits*

i.e., $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$.

4. *The limit of the product of two functions is equal to the product of their limits*

i.e., $\lim_{x \to a} [f(x) . g(x)] = \lim_{x \to a} f(x) . \lim_{x \to a} g(x)$.

5. *The limit of the quotient of two functions is equal to the quotients of their limits*

i.e., $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$.

6. *The limit of the product of constant and the function is equal to the product of the constant and limit of the function*

i.e., $\lim_{x \to a} [c f(x)] = c \lim_{x \to a} f(x)$, where c is a constant.

7. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$, provided $\sqrt[n]{\lim_{x \to a} f(x)}$ is a real number.

8. $\lim_{x \to a} [f(x)]^{p/q} = \left[\lim_{x \to a} f(x)\right]^{p/q}$.

Remark:

We know that if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, and $\lim_{x \to a} g(x) \neq 0$, then

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then $\frac{f(x)}{g(x)}$ is said to assume **indeterminate form** $\frac{0}{0}$ as $x \to a$. Other indeterminate forms are $\frac{\infty}{\infty}$, $\infty - \infty$, $0 . \infty$ etc.

9.3.1. Limit of a Polynomial Function

A function $f(x)$ defined for all real values of x is said to be *polynomial function* if $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$, where $a_n \neq 0$ for $n \in N$.

$a_0, a_1, a_2, ..., a_n$ are called its *coefficients* and are real numbers.

We know that $\lim_{x \to a} x = a$

$$\therefore \quad \lim_{x \to a} x^2 = \lim_{x \to a} (x.x) = \left(\lim_{x \to a} x\right).\left(\lim_{x \to a} x\right) = a.a = a^2$$

$$\lim_{x \to a} x^3 = \lim_{x \to a} (x.x.x)$$

$$= \left(\lim_{x \to a} x\right).\left(\lim_{x \to a} x\right).\left(\lim_{x \to a} x\right)$$

$$= a.a.a = a^3$$

....

....

$$\lim_{x \to a} x^n = a^n$$

Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$ be a polynomial function, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} [a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n]$$

$$= \lim_{x \to a} a_0 + \lim_{x \to a} a_1x + \lim_{x \to a} a_2x^2 + \ldots\ldots + \lim_{x \to a} a_nx^n$$

$$= a_0 + a_1 \lim_{x \to a} x + a_2 \lim_{x \to a} x^2 + \ldots\ldots + a_n \lim_{x \to a} x^n$$

$$= a_0 + a_1a + a_2a^2 + \ldots\ldots + a_na^n$$

$$= \boldsymbol{f(a)}.$$

9.3.2. Methods to Evaluate Limit of a Rational Function

A function $f(x)$ is said to be a rational function, if $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials such that $h(x) \neq 0$.

Let $f(x) = \dfrac{g(x)}{h(x)}$ so that $f(a) = \dfrac{g(a)}{h(a)}$

Case I. *When* $\dfrac{g(a)}{h(a)}$ *is neither* $\dfrac{0}{0}$ *form nor* $\dfrac{\infty}{\infty}$ *form.*

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)} = \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)}$$

$$= \frac{g(a)}{h(a)}\left[\text{provided } \frac{g(a)}{h(a)} \text{ is not } \frac{0}{0} \text{ or } \frac{\infty}{\infty} \text{ form}\right] = f(a)$$

$$\left[\because f(a) = \frac{g(a)}{h(a)}\right]$$

Thus, if $f(a)$ is neither $\dfrac{0}{0}$ nor $\dfrac{\infty}{\infty}$ form, then limit when $x \to a$ is obtained simply by putting $x = a$.

Case II. *When* $f(a) = \dfrac{g(a)}{h(a)} = \dfrac{0}{0}$ *form, then the following methods can be used :*

First Method. (*Method of factors*)

Factorise the numerator and denominator of $f(x)$ and cancel the common factors $(x - a) \neq 0$ [$\because x \to a \Rightarrow x \neq a \Rightarrow x - a \neq 0$] and when $f(a)$ becomes free of $\dfrac{0}{0}$ form, put $x = a$.

Second Method. (*Method of Substitution*)

To find $\lim\limits_{x \to a} f(x)$ **:** Put $x = a + h$, where h is a negligibly small number.

Now when $x \to a$, $h \to 0$. $\quad \therefore \lim\limits_{x \to a} f(x) = \lim\limits_{h \to 0} f(a + h)$

Now simplify $f(a + h)$ till it is free from $\dfrac{0}{0}$ form and then put $h = 0$.

Third Method. (*Method of Rationalisation*). This method can be used in cases of irrational functions, such as $\dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}$, $\dfrac{x}{1 - \sqrt{1-x}}$ etc.

SOLVED EXAMPLES

EXAMPLE 1. *Evaluate the limits :*

(*i*) $\lim\limits_{x \to 4} [x^2(x + 1)]$ $\qquad$ (*ii*) $\lim\limits_{x \to -1} [x^5 + x^4 + x^3 + x^2 + x + 1]$

Solution. Clearly, the required limits are limits of polynomial functions. Hence the limits are the values of the function at the specific points. Thus, we have

(*i*) $\lim\limits_{x \to 4} [x^2(x + 1)] = 4^2(4 + 1) = 16 \times 5 = \mathbf{80}.$

(*ii*) $\lim\limits_{x \to -1} [x^5 + x^4 + x^3 + x^2 + x + 1] = (-1)^5 + (-1)^4 + (-1)^3 + (-1)^2 + (-1) + 1$

$$= -1 + 1 - 1 + 1 - 1 + 1 = \mathbf{0}.$$

EXAMPLE 2. *Evaluate the limits :*

(*i*) $\lim\limits_{x \to 2} \dfrac{x^2 + 3x + 5}{x + 2}$ $\qquad$ (*ii*) $\lim\limits_{x \to -1} \dfrac{x^{10} + x^5 + 1}{x - 1}$ $\qquad$ (*iii*) $\lim\limits_{x \to -2} \dfrac{\sqrt{x^2 - 3}}{1 + x + x^2 + x^3}$

Solution. Clearly, the given functions are rational functions. Hence we can evaluate the limits of these functions at the specified points, provided the result is not of the form $\dfrac{0}{0}$.

(*i*) $$\lim_{x \to 2} \frac{x^2 + 3x + 5}{x + 2} = \frac{(2)^2 + 3(2) + 5}{2 + 2} = \frac{4 + 6 + 5}{4} = \mathbf{\frac{15}{4}}.$$

(ii) $$\lim_{x \to -1} \frac{x^{10} + x^5 + 1}{x - 1} = \frac{(-1)^{10} + (-1)^5 + 1}{(-1) - 1} = \frac{1 - 1 + 1}{-1 - 1} = -\frac{1}{2}.$$

(iii) $$\lim_{x \to -2} \frac{\sqrt{x^2 - 3}}{1 + x + x^2 + x^3} = \frac{\lim_{x \to -2} \sqrt{x^2 - 3}}{\lim_{x \to -2} (1 + x + x^2 + x^3)}$$

$$= \frac{\sqrt{(-2)^2 - 3}}{1 + (-2) + (-2)^2 + (-2)^3} = \frac{\sqrt{4 - 3}}{1 - 2 + 4 - 8} = -\frac{1}{5}.$$

EXAMPLE 3. *Evaluate :*

(i) $\lim_{x \to 3} \dfrac{x^4 - 81}{2x^2 - 5x - 3}$ (ii) $\lim_{x \to 1} \dfrac{(2x - 3)(\sqrt{x} - 1)}{2x^2 + x - 3}$

(iii) $\lim_{x \to 1} \left[\dfrac{x - 2}{x^2 - x} - \dfrac{1}{x^3 - 3x^2 + 2x} \right]$ (iv) $\lim_{x \to 1} \dfrac{x^2 - 4x + 3}{x^2 + 2x - 3}$

Solution. Here all the given functions are rational functions. Hence, we first evaluate the limit of these functions at the specified points. If the result is in the form $\frac{0}{0}$, then we shall rewrite the function by cancelling the factors due to which the limit takes the indeterminate form.

(i) $$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = \frac{(3)^4 - 81}{2(3)^2 - 5(3) - 3} = \frac{81 - 81}{18 - 15 - 3} = \frac{0}{0}$$

Thus, at $x = 3$, the given expression assumes the indeterminate form $\frac{0}{0}$.

Now, $$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = \frac{(x^2 - 9)(x^2 + 9)}{(x - 3)(2x + 1)}$$

$$= \lim_{x \to 3} \frac{(x - 3)(x + 3)(x^2 + 9)}{(x - 3)(2x + 1)} = \lim_{x \to 3} \frac{(x + 3)(x^2 + 9)}{(2x + 1)}$$

$$= \frac{(3 + 3)(9 + 9)}{(6 + 1)} = \frac{6 \times 18}{7} = \frac{108}{7}.$$

(ii) $$\lim_{x \to 1} \frac{(2x - 3)(\sqrt{x} - 1)}{2x^2 + x - 3} = \frac{(2 - 3)(1 - 1)}{2 + 1 - 3} = \frac{0}{0}$$

Thus, at $x = 1$, the given expression assumes the indeterminate form $\frac{0}{0}$.

Now, $$\lim_{x \to 1} \frac{(2x - 3)(\sqrt{x} - 1)}{2x^2 + x - 3} = \lim_{x \to 1} \frac{(2x - 3)(\sqrt{x} - 1)}{(2x + 3)(x - 1)}$$

$$= \lim_{x \to 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(\sqrt{x}-1)(\sqrt{x}+1)}$$

$$= \lim_{x \to 1} \frac{2x-3}{(2x+3)(\sqrt{x}+1)} \quad [\because x \to 1 \Rightarrow \sqrt{x} \neq 1 \Rightarrow \sqrt{x}-1 \neq 0]$$

$$= \frac{2-3}{(2+3)(2)} = -\frac{1}{10}.$$

(*iii*) Let us first rewrite the given function as a rational function.

$$\frac{x-2}{x^2-x} - \frac{1}{x^3-3x^2+2x} = \frac{x-2}{x(x-1)} - \frac{1}{x(x^2-3x+2)}$$

$$= \frac{x-2}{x(x-1)} - \frac{1}{x(x-1)(x-2)}$$

$$= \frac{(x-2)^2-1}{x(x-1)(x-2)} = \frac{x^2-4x+4-1}{x(x-1)(x-2)}$$

$$= \frac{x^2-4x+3}{x(x-1)(x-2)}$$

Hence, $$\lim_{x \to 1} \left[\frac{x-2}{x^2-x} - \frac{1}{x^3-3x^2+2x}\right] = \lim_{x \to 1} \frac{x^2-4x+3}{x(x-1)(x-2)} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 1} \frac{(x-1)(x-3)}{x(x-1)(x-2)} = \lim_{x \to 1} \frac{x-3}{x(x-2)}$$

$$= \frac{1-3}{1(1-2)} = \frac{-2}{-1} = \mathbf{2}.$$

(*iv*) When $x = 1$, the given expression assumes the indeterminate form $\frac{0}{0}$.

[*Using substitution method*]

Put $x = 1 + h$. Now as $x \to 1, h \to 0$

$$\therefore \quad \lim_{x \to 1} \frac{x^2-4x+3}{x^2+2x-3} = \lim_{h \to 0} \frac{(1+h)^2-4(1+h)+3}{(1+h)^2+2(1+h)-3}$$

$$= \lim_{h \to 0} \frac{1+h^2+2h-4-4h+3}{1+h^2+2h+2+2h-3}$$

$$= \lim_{h \to 0} \frac{h^2-2h}{h^2+4h} = \lim_{h \to 0} \frac{h(h-2)}{h(h+4)}$$

$$= \lim_{h \to 0} \frac{h-2}{h+4} = \frac{0-2}{0+4} = -\frac{2}{4} = -\frac{1}{2}.$$

EXAMPLE 4. *Evaluate :*

(*i*) $\lim_{x \to 2} \frac{x^2-4}{x-2}$ (*ii*) $\lim_{x \to 4} \frac{x^3-64}{x^2-16}$ (*iii*) $\lim_{x \to 3} \frac{x^3-4x-15}{x^3+x^2-6x-18}$.

Solution. (*i*) **First Method** (*By Factors*)

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2}$$

$$= \lim_{x \to 2} (x + 2) = 2 + 2 = \mathbf{4}. \qquad [\because x \to 2 \text{ means } x - 2 \neq 0]$$

Second Method (*By Substitution*)

Put $x = 2 + h$. Now as $x \to 2$, $h \to 0$

$$\therefore \quad \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{h \to 0} \frac{(2+h)^2 - 4}{2 + h - 2} = \lim_{h \to 0} \frac{4 + h^2 + 4h - 4}{h}$$

$$= \lim_{h \to 0} \frac{h(h+4)}{h} = \lim_{h \to 0} (h + 4) = 0 + 4 = \mathbf{4}.$$

(*ii*) When $x = 4$, the given expression assumes the indeterminate form $\frac{0}{0}$.

Now, $$\lim_{x \to 4} \frac{x^3 - 64}{x^2 - 16} = \lim_{x \to 4} \frac{(x-4)(x^2 + 4x + 16)}{(x-4)(x+4)}$$

[*Dividing the numerator and denominator by $x - 4$ as $x \to 4 \Rightarrow x \neq 4$*]

$$= \lim_{x \to 4} \frac{x^2 + 4x + 16}{x + 4} = \frac{(4)^2 + 4(4) + 16}{4 + 4}$$

$$= \frac{16 + 16 + 16}{8} = \frac{48}{8} = \mathbf{6}.$$

(*iii*) When $x = 3$, the given expression assumes the indeterminate form $\frac{0}{0}$

Now, $$\lim_{x \to 3} \frac{x^3 - 4x - 15}{x^3 + x^2 - 6x - 18} = \lim_{x \to 3} \frac{(x-3)(x^2 + 3x + 5)}{(x-3)(x^2 + 4x + 6)}$$

[*Dividing the numerator and denominator by $(x - 3)$ as $x \to 3 \Rightarrow x - 3 \neq 0$*]

$$= \lim_{x \to 3} \frac{x^2 + 3x + 5}{x^2 + 4x + 6} = \frac{9 + 9 + 5}{9 + 12 + 6} = \mathbf{\frac{23}{27}}.$$

EXAMPLE 5. *Evaluate :*

(*i*) $\lim_{x \to 2} \left(\frac{1}{x - 2} - \frac{4}{x^3 - 2x^2} \right)$ (*ii*) $\lim_{x \to 3} \left[\frac{1}{x - 3} - \frac{3}{x(x^2 - 5x + 6)} \right]$

Solution. (*i*) $$\lim_{x \to 2} \left(\frac{1}{x - 2} - \frac{4}{x^3 - 2x^2} \right) = \lim_{x \to 2} \left(\frac{1}{x - 2} - \frac{4}{x^2(x - 2)} \right)$$

$$= \lim_{x \to 2} \frac{x^2 - 4}{x^2(x-2)} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x^2(x-2)}$$

$$= \lim_{x \to 2} \frac{x+2}{x^2} = \frac{2+2}{4} = \frac{4}{4} = \mathbf{1}.$$

(*ii*) $\lim_{x \to 3} \left[\frac{1}{x-3} - \frac{3}{x(x^2-5x+6)}\right] = \lim_{x \to 3} \left[\frac{1}{x-3} - \frac{3}{x(x-2)(x-3)}\right]$

$$= \lim_{x \to 3} \frac{x(x-2)-3}{x(x-2)(x-3)} = \lim_{x \to 3} \frac{x^2-2x-3}{x(x-2)(x-3)}$$

$$= \lim_{x \to 3} \frac{(x+1)(x-3)}{x(x-2)(x-3)} = \lim_{x \to 3} \frac{x+1}{x(x-2)}$$

$$= \frac{3+1}{3(3-2)} = \mathbf{\frac{4}{3}}.$$

EXAMPLE 6. *Evaluate* :

(*i*) $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{2x}$ (*ii*) $\lim_{x \to 4} \frac{3 - \sqrt{5+x}}{1 - \sqrt{5-x}}$

Solution. (*i*) When $x = 0$, the given expression assumes the indeterminate form $\frac{0}{0}$.

Now, $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{2x} = \lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{2x} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$ [*Rationalising*]

$$= \lim_{x \to 0} \frac{(1+x)-(1-x)}{2x(\sqrt{1+x} + \sqrt{1-x})} = \lim_{x \to 0} \frac{2x}{2x(\sqrt{1+x} + \sqrt{1-x})}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{1+x} + \sqrt{1-x}} = \mathbf{\frac{1}{2}}.$$

(*ii*) When $x = 4$, the given expression assumes the indeterminate form $\frac{0}{0}$

Now, $\lim_{x \to 4} \frac{3 - \sqrt{5+x}}{1 - \sqrt{5-x}} = \lim_{x \to 4} \left(\frac{3 - \sqrt{5+x}}{1 - \sqrt{5-x}} \times \frac{1 + \sqrt{5-x}}{1 + \sqrt{5-x}} \times \frac{3 + \sqrt{5+x}}{3 + \sqrt{5+x}}\right)$ [*Rationalising*]

$$= \lim_{x \to 4} \left(\frac{9-(5+x)}{1-(5-x)} \times \frac{1+\sqrt{5-x}}{3+\sqrt{5+x}}\right)$$

$$= \lim_{x \to 4} \left(\frac{4-x}{x-4} \times \frac{1+\sqrt{5-x}}{3+\sqrt{5+x}}\right) = \lim_{x \to 4} \left((-1) \times \frac{1+\sqrt{5-x}}{3+\sqrt{5+x}}\right)$$

$$= -\frac{1+1}{3+3} = -\frac{2}{6} = -\mathbf{\frac{1}{3}}.$$

EXERCISE 9.1

1. Evaluate the following limits :

(i) $\lim_{x \to 1} [x^3 - x^2 + 1]$

(ii) $\lim_{x \to -1} [1 + x + x^2 + \dots + x^{10}]$

(iii) $\lim_{x \to 1} [(x-1)^2 + 5]$

(iv) $\lim_{x \to 0} (10^n x + 1), n \in \mathbf{N}$

2. Evaluate :

(i) $\lim_{x \to 1} \frac{x^2+1}{x+100}$

(ii) $\lim_{x \to 0} \frac{x^4 - 3x^3 + 2}{x^3 - 5x^2 + 3x + 1}$

(iii) $\lim_{x \to 3} \frac{x^2 - 9}{x+2}$

(iv) $\lim_{x \to 0} \frac{\sqrt{1+x} + \sqrt{1-x}}{1+x}$

3. Evaluate :

(i) $\lim_{x \to \frac{1}{2}} \frac{4x^2 - 1}{2x - 1}$

(ii) $\lim_{x \to 2} \frac{x^3 - 4x^2 + 4x}{x^2 - 4}$

(iii) $\lim_{x \to 2} \frac{x^3 - 2x^2}{x^2 - 5x + 6}$

(iv) $\lim_{x \to 5} \frac{x^2 - 9x + 20}{x^2 - 6x + 5}$

(v) $\lim_{x \to -3} \frac{x^3 + 4x^2 + 4x + 3}{x^2 + 2x - 3}$.

(vi) $\lim_{x \to -\frac{2}{3}} \frac{3x^2 + 5x + 2}{3x^2 - 4x - 4}$

4. Evaluate :

(i) $\lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$

(ii) $\lim_{x \to \sqrt{2}} \frac{x^2 - 2}{x^2 + \sqrt{2}x - 4}$

(iii) $\lim_{x \to 4} \frac{(x^2 - x - 12)^{15}}{(x^3 - 8x^2 + 16x)^7}$

(iv) $\lim_{x \to \sqrt{3}} \frac{x^4 - 9}{x^2 + 4\sqrt{3}x - 15}$

5. Evaluate :

(i) $\lim_{x \to 2} \left(\frac{4}{x^2 - 4} + \frac{1}{2 - x} \right)$

(ii) $\lim_{x \to 3} (x^2 - 9) \left(\frac{1}{x+3} + \frac{1}{x-3} \right)$

(iii) $\lim_{x \to 1} \left(\frac{1}{x^2 + x - 2} - \frac{x}{x^3 - 1} \right)$

(iv) $\lim_{x \to 2} \left(\frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 3x^2 + 2x} \right)$

6. Evaluate :

(i) $\lim_{x \to 0} \frac{x}{1 - \sqrt{1-x}}$

(ii) $\lim_{x \to 0} \frac{\sqrt{1+3x} - \sqrt{1-3x}}{x}$

(iii) $\lim_{x \to 3} \frac{x^2 - 9}{\sqrt{3x-2} - \sqrt{x+4}}$

(iv) $\lim_{x \to 0} \frac{\sqrt{a+x} - \sqrt{a}}{x\sqrt{a(a+x)}}$

ANSWERS

1. (i) 1 (ii) 1 (iii) 5 (iv) 1
2. (i) $\frac{2}{101}$ (ii) 2 (iii) 0 (iv) 2
3. (i) 2 (ii) 0 (iii) -4 (iv) $\frac{1}{4}$ (v) $-\frac{7}{4}$ (vi) $-\frac{1}{8}$
4. (i) $\frac{1}{2\sqrt{a}}$ (ii) $\frac{2}{3}$ (iii) 0 (iv) 2
5. (i) $-\frac{1}{4}$ (ii) 6 (iii) $-\frac{1}{9}$ (iv) $-\frac{1}{2}$
6. (i) 2 (ii) 3 (iii) $6\sqrt{7}$ (iv) $\frac{1}{2\,a^{3/2}}$

9.4. ▶ A SPECIAL LIMIT

For any positive integer n, $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.

Proof. Dividing $(x^n - a^n)$ by $(x - a)$, we get

$$x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + \dots + a^{n-2}x + a^{n-1})$$

$$\therefore \quad \lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} [x^{n-1} + ax^{n-2} + \dots + a^{n-2}x + a^{n-1}]$$

$$= a^{n-1} + a.a^{n-2} + \dots + a^{n-2}.a + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + \dots + a^{n-1} + a^{n-1} \text{ (} n \text{ terms)}$$

$$= na^{n-1}$$

Hence, $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

Note:

1. *The expression in the above theorem for the limit is true even if n is any rational number and a is positive.*
2. *The result of this limit can be used directly in the problems.*

SOLVED EXAMPLES

EXAMPLE 1. *Evaluate :*

(i) $\lim\limits_{x \to 3} \frac{x^5 - 243}{x^2 - 9}$ (ii) $\lim\limits_{x \to a} \frac{x\sqrt{x} - a\sqrt{a}}{x - a}$

(iii) $\lim\limits_{x \to 2} \frac{x^{1/4} - 2^{1/4}}{x^{1/3} - 2^{1/3}}$ (iv) $\lim\limits_{x \to a} \frac{(x + 2)^{3/2} - (a + 2)^{3/2}}{x - a}$

Solution. (*i*) $$\lim_{x\to 3}\frac{x^5-243}{x^2-9}=\lim_{x\to 3}\frac{\dfrac{x^5-243}{x-3}}{\dfrac{x^2-9}{x-3}}=\frac{\lim\limits_{x\to 3}\dfrac{x^5-3^5}{x-3}}{\lim\limits_{x\to 3}\dfrac{x^2-3^2}{x-3}}$$

$$=\frac{5.(3)^{5-1}}{2.(3)^{2-1}}=\frac{5.81}{2.3}=\frac{\mathbf{135}}{\mathbf{2}}. \qquad \left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

(*ii*) $$\lim_{x\to a}\frac{x\sqrt{x}-a\sqrt{a}}{x-a}=\lim_{x\to a}\frac{x^{3/2}-a^{3/2}}{x-a}=\frac{3}{2}a^{3/2-1}$$

$$=\frac{3}{2}a^{1/2}=\frac{\mathbf{3}}{\mathbf{2}}\sqrt{\boldsymbol{a}}.$$

(*iii*) $$\lim_{x\to 2}\frac{x^{1/4}-2^{1/4}}{x^{1/3}-2^{1/3}}=\lim_{x\to 2}\frac{\dfrac{x^{1/4}-2^{1/4}}{x-2}}{\dfrac{x^{1/3}-2^{1/3}}{x-2}}=\frac{\dfrac{1}{4}(2)^{\frac{1}{4}-1}}{\dfrac{1}{3}(2)^{\frac{1}{3}-1}}$$

$$=\frac{3}{4}(2)^{\frac{1}{4}-1-\frac{1}{3}+1}=\frac{3}{4}(2)^{\frac{1}{4}-\frac{1}{3}}=\frac{\mathbf{3}}{\mathbf{4}}.\mathbf{2^{-1/12}}.$$

(*iv*) $$\lim_{x\to a}\frac{(x+2)^{3/2}-(a+2)^{3/2}}{x-a}=\lim_{(x+2)\to(a+2)}\frac{(x+2)^{3/2}-(a+2)^{3/2}}{(x+2)-(a+2)}$$

$$=\frac{3}{2}(a+2)^{\frac{3}{2}-1}=\frac{\mathbf{3}}{\mathbf{2}}\boldsymbol{(a+2)^{1/2}}.$$

EXAMPLE 2. *Evaluate*

(*i*) $\displaystyle\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$ (*ii*) $\displaystyle\lim_{x\to 0}\frac{(1+x)^6-1}{(1+x)^2-1}$

Solution. (*i*) Put $y=1+x$, so that as $x\to 0$, $y\to 1$.

$$\therefore \qquad \lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{y\to 1}\frac{\sqrt{y}-1}{y-1}$$

$$=\lim_{y\to 1}\frac{y^{\frac{1}{2}}-1^{\frac{1}{2}}}{y-1}=\frac{1}{2}(1)^{\frac{1}{2}-1} \qquad \left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$=\frac{1}{2}\times(1)^{-\frac{1}{2}}=\frac{1}{2}\times 1=\frac{\mathbf{1}}{\mathbf{2}}.$$

(*ii*) Put $1 + x = y$, so that as $x \to 0, y \to 1$

$$\therefore \quad \lim_{x \to 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1} = \lim_{y \to 1} \frac{y^6 - 1}{y^2 - 1}$$

$$= \frac{\lim\limits_{y \to 1} \dfrac{y^6 - 1}{y - 1}}{\lim\limits_{y \to 1} \dfrac{y^2 - 1}{y - 1}} = \frac{6(1)^{6-1}}{2(1)^{2-1}} = \frac{6}{2} = \mathbf{3}. \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

Alternative method : Put $(1 + x)^2 = y$. Now as $x \to 0, y \to 1$

$$\therefore \quad \lim_{x \to 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1} = \lim_{y \to 1} \frac{y^3 - 1}{y - 1} = \lim_{y \to 1} \frac{(y-1)(y^2 + y + 1)}{y - 1}$$

$$= \lim_{y \to 1} (y^2 + y + 1) = 1 + 1 + 1 = \mathbf{3}.$$

EXAMPLE 3. *If* $\lim\limits_{x \to -a} \dfrac{x^9 + a^9}{x + a} = 9$, *find the values of a.*

Solution. $\lim\limits_{x \to -a} \dfrac{x^9 + a^9}{x + a} = 9$

$$\therefore \quad \lim_{x \to -a} \frac{x^9 - (-a^9)}{x - (-a)} = 9 \Rightarrow 9(-a)^{9-1} = 9 \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$\Rightarrow \quad 9a^8 = 9 \Rightarrow a^8 = 1 \Rightarrow a = \pm \mathbf{1}.$$

EXAMPLE 4. *If* $\lim\limits_{x \to 1} \dfrac{x^4 - 1}{x - 1} = \lim\limits_{x \to k} \dfrac{x^3 - k^3}{x^2 - k^2}$, *find the value of k.*

Solution. $\lim\limits_{x \to 1} \dfrac{x^4 - 1}{x - 1} = \lim\limits_{x \to k} \dfrac{x^3 - k^3}{x^2 - k^2}$

$$\Rightarrow \quad \lim_{x \to 1} \frac{x^4 - 1^4}{x - 1} = \lim_{x \to k} \frac{\dfrac{x^3 - k^3}{x - k}}{\dfrac{x^2 - k^2}{x - k}}$$

$$\Rightarrow \quad \lim_{x \to 1} \frac{x^4 - 1^4}{x - 1} = \frac{\lim\limits_{x \to k} \dfrac{x^3 - k^3}{x - k}}{\lim\limits_{x \to k} \dfrac{x^2 - k^2}{x - k}}$$

$$\Rightarrow \quad 4(1)^3 = \frac{3(k)^2}{2(k)^1} \quad \left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$\Rightarrow \quad 4 = \frac{3}{2}k \Rightarrow k = \mathbf{\frac{8}{3}}.$$

EXERCISE 9.2

1. Evaluate :

(i) $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$ (ii) $\lim\limits_{x \to 2} \dfrac{x^{10} - 1024}{x - 2}$

(iii) $\lim\limits_{x \to 1} \dfrac{x^{15} - 1}{x^{10} - 1}$

2. Evaluate :

(i) $\lim\limits_{x \to 0} \dfrac{(1 + x)^n - 1}{x}$ (ii) $\lim\limits_{x \to a} \dfrac{x^{2/3} - a^{2/3}}{x^{3/4} - a^{3/4}}$

(iii) $\lim\limits_{x \to a} \dfrac{(x + 2)^{5/3} - (a + 2)^{5/3}}{x - a}$ (iv) $\lim\limits_{x \to 1} \dfrac{1 - x^{-1/3}}{1 - x^{-2/3}}$

3. If $\lim\limits_{x \to a} \dfrac{x^9 - a^9}{x - a} = 9$, find all possible values of a.

4. If $\lim\limits_{x \to 2} \dfrac{x^n - 2^n}{x - 2} = 80$ and if n is a positive integer, find n.

5. If $\lim\limits_{x \to a} \dfrac{x^5 - a^5}{x - a} = 405$, find all possible values of a.

ANSWERS

1. (i) 3 (ii) 5120 (iii) $\dfrac{3}{2}$

2. (i) n (ii) $\dfrac{8}{9} a^{-1/12}$ (iii) $\dfrac{5}{3}(a + 2)^{2/3}$ (iv) $\dfrac{1}{2}$

3. $a = \pm 1$ 4. $n = 5$ 5. $a = \pm 3$

9.5. ▶ LIMITS INVOLVING EXPONENTIAL AND LOGARITHMIC FUNCTIONS

We have already discussed exponential and logarithmic functions in the previous chapter. We now state some important standard limits (without proof) involving exponential and logarithmic functions.

(i) $\lim\limits_{x \to 0} (1 + x)^{1/x} = e$ **(ii)** $\lim\limits_{x \to 0} \dfrac{\log_e (1 + x)}{x} = 1$

(iii) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$ **(iv)** $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log a$

SOLVED EXAMPLES

EXAMPLE 1. *Prove that :*

(i) $\lim\limits_{x \to 1} e^x = e$ (ii) $\lim\limits_{x \to e} \log x = 1.$

Solution. (i) Put $x = 1 + h$. Now as $x \to 1, h \to 0$

$$\therefore \quad \lim_{x \to 1} e^x = \lim_{h \to 0} e^{1+h} = \lim_{h \to 0} e^1 . e^h$$

$$= e \lim_{h \to 0} e^h = e . 1 = \mathbf{e}.$$

(ii) Put $y = \dfrac{x}{e}$. Now as $x \to e, y \to 1$

$$\therefore \quad \lim_{x \to e} \log x = \lim_{y \to 1} \log ey$$

$$= \lim_{y \to 1} (\log e + \log y)$$

$$= 1 + \lim_{y \to 1} \log y = 1 + \log 1 = 1 + 0 = \mathbf{1}.$$

EXAMPLE 2. *Evaluate* $\lim\limits_{x \to 1} \dfrac{\log_e x}{x - 1}$.

Solution. Put $x - 1 = y$; $\therefore \quad x = 1 + y$. Now as $x \to 1, y \to 0$.

$$\therefore \quad \lim_{x \to 1} \frac{\log_e x}{x - 1} = \lim_{y \to 0} \frac{\log_e (1 + y)}{y} = \mathbf{1}.$$

EXAMPLE 3. *Evaluate :*

(i) $\lim\limits_{x \to 0} \dfrac{a^x - b^x}{x}$ (ii) $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{x}$ (iii) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{b^x - 1}$.

Solution. (i) $\lim\limits_{x \to 0} \dfrac{a^x - b^x}{x} = \lim\limits_{x \to 0} \dfrac{(a^x - 1) - (b^x - 1)}{x}$

$$= \lim_{x \to 0} \frac{a^x - 1}{x} - \lim_{x \to 0} \frac{b^x - 1}{x}$$

$$= \log a - \log b = \mathbf{\log \frac{a}{b}}.$$

(ii) $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{x} = \lim\limits_{x \to 0} \dfrac{(e^x - 1) - (e^{-x} - 1)}{x}$

$$= \lim_{x \to 0} \frac{e^x - 1}{x} - \lim_{x \to 0} \frac{e^{-x} - 1}{x}$$

$$= \lim_{x \to 0} \frac{e^x - 1}{x} + \lim_{-x \to 0} \frac{e^{-x} - 1}{-x} \qquad [\because \text{ As } x \to 0, -x \to 0]$$

$$= 1 + 1 = \mathbf{2}.$$

(iii) $$\lim_{x \to 0} \frac{a^x - 1}{b^x - 1} = \lim_{x \to 0} \left(\frac{a^x - 1}{x} \cdot \frac{x}{b^x - 1} \right) = \frac{\lim\limits_{x \to 0} \dfrac{a^x - 1}{x}}{\lim\limits_{x \to 0} \dfrac{b^x - 1}{x}}$$

$$= \frac{\log a}{\log b}.$$

Note: *If no base of log is mentioned, then it is taken for granted that the base is e. Thus log a is same as $\log_e a$.*

EXAMPLE 4. *Evaluate* $\lim\limits_{x \to 0} \dfrac{e^{2+x} - e^2}{x}$

Solution. $$\lim_{x \to 0} \frac{e^{2+x} - e^2}{x} = \lim_{x \to 0} \frac{e^2 . e^x - e^2}{x} = \lim_{x \to 0} \frac{e^2 (e^x - 1)}{x}$$

$$= e^2 . \lim_{x \to 0} \frac{e^x - 1}{x} = e^2 . 1 = \mathbf{e^2}.$$

EXAMPLE 5. *Evaluate* $\lim\limits_{x \to 5} \dfrac{e^x - e^5}{x - 5}$.

Solution. Put $x = 5 + h$. Now as $x \to 5$, $h \to 0$

$$\therefore \quad \lim_{x \to 5} \frac{e^x - e^5}{x - 5} = \lim_{h \to 0} \frac{e^{5+h} - e^5}{5 + h - 5}$$

$$= \lim_{h \to 0} \frac{e^5 . e^h - e^5}{h} = \lim_{h \to 0} e^5 \left(\frac{e^h - 1}{h} \right) = e^5 . 1 = \mathbf{e^5}.$$

EXAMPLE 6. *Evaluate the following limits :*

(i) $\lim\limits_{x \to 2} \dfrac{\log x - \log 2}{x - 2}$ (ii) $\lim\limits_{x \to 0} \dfrac{\log (a + x) - \log (a - x)}{x}$, $a > 0$

Solution. (i) Put $x = 2 + h$. Now when $x \to 2$, $h \to 0$

$$\therefore \quad \lim_{x \to 2} \frac{\log x - \log 2}{x - 2} = \lim_{h \to 0} \frac{\log (2 + h) - \log 2}{2 + h - 2} = \lim_{h \to 0} \frac{\log \left(\dfrac{2 + h}{2} \right)}{h}$$

$$= \lim_{h \to 0} \frac{\log \left(1 + \dfrac{h}{2} \right)}{\dfrac{h}{2}} . \frac{1}{2} = 1 . \frac{1}{2} = \frac{\mathbf{1}}{\mathbf{2}}.$$

(ii) $\lim\limits_{x \to 0} \dfrac{\log(a+x) - \log(a-x)}{x} = \lim\limits_{x \to 0} \dfrac{\log\left[a\left(1+\frac{x}{a}\right)\right] - \log\left[a\left(1-\frac{x}{a}\right)\right]}{x}$

$$= \lim_{x \to 0} \frac{\left[\log a + \log\left(1+\frac{x}{a}\right)\right] - \left[\log a + \log\left(1-\frac{x}{a}\right)\right]}{x}$$

$$= \lim_{x \to 0} \frac{\log\left(1+\frac{x}{a}\right) - \log\left(1-\frac{x}{a}\right)}{x}$$

$$= \left(\lim_{x \to 0} \frac{1}{a} \cdot \frac{\log\left(1+\frac{x}{a}\right)}{x/a}\right) - \left(\lim_{x \to 0} \frac{1}{(-a)} \cdot \frac{\log\left(1-\frac{x}{a}\right)}{-x/a}\right)$$

$$= \frac{1}{a} + \frac{1}{a} = \frac{2}{a}.$$

EXERCISE 9.3

1. Evaluate :

(i) $\lim\limits_{x \to 0} (1 + 2x)^{1/x}$ (ii) $\lim\limits_{x \to 0} (1 + bx)^{1/x}$ (iii) $\lim\limits_{x \to 0} (1 + 3x)^{1/4x}$

2. Evaluate :

(i) $\lim\limits_{x \to 0} \dfrac{3^x - 2^x}{x}$ (ii) $\lim\limits_{y \to 0} \dfrac{a^y - a^{-y}}{y}$ (iii) $\lim\limits_{x \to 0} \dfrac{e^{3x} - 1}{x}$

(iv) $\lim\limits_{x \to 0} \dfrac{5^x - 1}{3^x - 1}$ (v) $\lim\limits_{x \to 0} \dfrac{7^{2x} - 1}{x}$ (vi) $\lim\limits_{x \to 0} \dfrac{3^{2+x} - 9}{x}$

3. Evaluate :

(i) $\lim\limits_{x \to 0} \dfrac{e^x - x - 1}{x}$ (ii) $\lim\limits_{x \to 2} \dfrac{e^x - e^2}{x - 2}$ (iii) $\lim\limits_{x \to -1} \dfrac{e^x - e^{-1}}{x + 1}$

(iv) $\lim\limits_{x \to 0} \dfrac{e^{bx} - e^{ax}}{x}$, $0 < a < b$ (v) $\lim\limits_{x \to 0} \dfrac{(ab)^x - a^x - b^x + 1}{x^2}$

4. Evaluate :

(i) $\lim\limits_{x \to 0} \dfrac{\log(1 + 3x)}{3^x - 1}$ (ii) $\lim\limits_{x \to 5} \dfrac{\log x - \log 5}{x - 5}$

(iii) $\lim\limits_{x \to 0} \dfrac{\log(a + x) - \log a}{x}$ (iv) $\lim\limits_{x \to 2} \dfrac{x - 2}{\log_a(x - 1)}$

ANSWERS

1. (i) e^2 (ii) e^b (iii) $e^{3/4}$

2. (i) $\log \frac{3}{2}$ (ii) $2 \log a$ (iii) 3 (iv) $\frac{\log 5}{\log 3}$ (v) $2 \log 7$ (vi) $9 \log 3$

3. (i) 0 (ii) e^2 (iii) $\frac{1}{e}$ (iv) $b - a$ (v) $(\log a)(\log b)$

4. (i) $\frac{3}{\log 3}$ (ii) $\frac{1}{5}$ (iii) $\frac{1}{a}$ (iv) $\log a$

9.6. ▶ EVALUATION OF LEFT HAND AND RIGHT HAND LIMITS

We are aware of the fact that the statement $x \to a^-$ means that x is approaching a from the left hand side *i.e.*, x is a number less than a but very close to a.

Similarly, $x \to a^+$ means that x is approaching a from the right hand side *i.e.*, x is a number greater than a but very close to a. We have already given a brief idea of left hand limits and right hand limits in earlier section which we define as follows :

9.6.1. Left hand limit of a function

Let $f(x)$ be a function of x. If $f(x) \to l$ as $x \to a^-$, then l is said to be the left hand limit of a function $f(x)$ at a and we write it as

$$\lim_{x \to a-0} f(x) = l \text{ or } \lim_{x \to a^-} f(x) = l.$$

9.6.2. Right hand limit of a function

Let $f(x)$ be a function of x. If $f(x) \to l$ as $x \to a^+$, then l is said to be the right hand limit of a function $f(x)$ at a and we write it as

$$\lim_{x \to a+0} f(x) = l \text{ or } \lim_{x \to a^+} f(x) = l.$$

Remark :

The limit of $f(x)$ as $x \to a$ exists, if and only if both right hand and left hand limits exist and are equal. Thus $\lim_{x \to a} f(x) = l$ if and only if $\lim_{x \to a+0} f(x) = l = \lim_{x \to a-0} f(x)$.

9.6.3. Working rule to calculate left hand limit and right hand limit.

(i) Left hand limit $= \lim_{x \to a-0} f(x)$

As $x \to a - 0$; put $x \to a - h$; where $h \to 0$

$\therefore$ L.H.L. $= \lim_{x \to a-0} f(x) = \lim_{h \to 0} f(a - h)$

(ii) Right hand limit $= \lim_{x \to a+0} f(x)$

As $x \to a + 0$; put $x = a + h$, where $h \to 0$

$\therefore$ R.H.L. $= \lim_{x \to a+0} f(x) = \lim_{h \to 0} f(a + h).$

SOLVED EXAMPLES

EXAMPLE 1. *Show that* $\lim\limits_{x \to 0} \frac{1}{x}$ *does not exist.*

Solution. L.H.L. $= \lim\limits_{x \to 0^-} \frac{1}{x} = \lim\limits_{h \to 0} \frac{1}{0-h} = -\lim\limits_{h \to 0} \frac{1}{h} = -\infty$ [*Putting* $x = 0 - h$]

R.H.L. $= \lim\limits_{x \to 0^+} \frac{1}{x} = \lim\limits_{h \to 0} \frac{1}{0+h} = \lim\limits_{h \to 0} \frac{1}{h} = \infty$ [*Putting* $x = 0 + h$]

Now, L.H.L. $\neq$ R.H.L.

Hence $\lim\limits_{x \to 0} \frac{1}{x}$ does not exist.

EXAMPLE 2. *Evaluate the following one sided limits :*

(*i*) $\lim\limits_{x \to 2^+} \frac{x-3}{x^2-4}$ (*ii*) $\lim\limits_{x \to (-2)^+} \left(\frac{x^2-1}{2x+4}\right)$

Solution. (*i*) $\lim\limits_{x \to 2^+} \frac{x-3}{x^2-4} = \lim\limits_{h \to 0} \frac{(2+h)-3}{(2+h)^2-4}$ [*Putting* $x = 2 + h$]

$$= \lim_{h \to 0} \frac{h-1}{(4+4h+h^2)-4} = \lim_{h \to 0} \frac{h-1}{4h+h^2} = -\frac{1}{0} = -\infty.$$

(*ii*) $\lim\limits_{x \to (-2)^+} \left(\frac{x^2-1}{2x+4}\right) = \lim\limits_{h \to 0} \frac{(-2+h)^2-1}{2(-2+h)+4}$ [*Putting* $x = -2 + h$]

$$= \lim_{h \to 0} \frac{(4-4h+h^2)-1}{(-4+2h)+4} = \lim_{h \to 0} \frac{3-4h+h^2}{2h}$$

$$= \frac{3-0+0}{2(0)} = \frac{3}{0} = \infty.$$

EXAMPLE 3. *Evaluate the following limits, if they exist :*

(*i*) $\lim\limits_{x \to 2} |x-2|$ (*ii*) $\lim\limits_{x \to 4} \sqrt{8-2x}$

Solution. (*i*) L.H.L. $= \lim\limits_{x \to 2^-} |x-2| = \lim\limits_{h \to 0} |2-h-2|$ [*Putting* $x = 2 - h$]

$$= \lim_{h \to 0} |-h| = \lim_{h \to 0} h = 0$$

R.H.L. $= \lim\limits_{x \to 2^+} |x-2| = \lim\limits_{h \to 0} |2+h-2|$ [*Putting* $x = 2 + h$]

$$= \lim_{h \to 0} |h| = \lim_{h \to 0} h = 0$$

Now, L.H.L. = R.H.L. = 0

Hence, $\lim\limits_{x \to 2} |x-2| = \mathbf{0}$.

(ii) L.H.L. $= \lim_{x \to 4^-} \sqrt{8-2x} = \lim_{h \to 0} \sqrt{8-2(4-h)}$ [*Putting* $x = 4 - h$]

$= \lim_{h \to 0} \sqrt{8-8+2h} = \lim_{h \to 0} \sqrt{2h} = 0$

R.H.L. $= \lim_{x \to 4^+} \sqrt{8-2x} = \lim_{h \to 0} \sqrt{8-2(4+h)}$ [*Putting* $x = 4 + h$]

$= \lim_{h \to 0} \sqrt{8-8-2h} = \lim_{h \to 0} \sqrt{-2h}$, which is not defined.

Hence, $\lim_{x \to 4} \sqrt{8-2x}$ does not exist.

EXAMPLE 4. *Determine whether the following limits exist :*

(i) $\lim_{x \to 0} \frac{x}{|x|}$ (ii) $\lim_{x \to 0} \frac{[x]}{x}$, *where* $[x]$ *denotes the greatest integer* $\le x$.

Solution. (i) L.H.L. $= \lim_{x \to 0-} \frac{x}{|x|} = \lim_{h \to 0} \frac{0-h}{|0-h|} = \frac{-h}{h} = -1$ [*Putting* $x = 0 - h$]

R.H.L. $= \lim_{x \to 0+} \frac{x}{|x|} = \lim_{h \to 0} \frac{0+h}{|0+h|} = \frac{h}{h} = 1$ [*Putting* $x = 0 + h$]

Now, L.H.L. $\neq$ R.H.L.

Hence, $\lim_{x \to 0} \frac{x}{|x|}$ does not exist.

(ii) L.H.L. $= \lim_{x \to 0^-} \frac{[x]}{x} = \lim_{h \to 0} \frac{[0-h]}{0-h}$ [*Putting* $x = 0 - h$]

$= \lim_{h \to 0} \frac{[-h]}{-h} = \lim_{h \to 0} \frac{-1}{-h} = \infty$

R.H.L. $= \lim_{x \to 0^+} \frac{[x]}{x} = \lim_{h \to 0} \frac{[0+h]}{0+h}$ [*Putting* $x = 0 + h$]

$= \lim_{h \to 0} \frac{[h]}{h} = \lim_{h \to 0} \frac{0}{h} = 0$

Now, L.H.L $\neq$ R.H.L. Hence $\lim_{x \to 0} \frac{[x]}{x}$ does not exist.

EXAMPLE 5. *Find* $\lim_{x \to \frac{5}{2}} [x]$.

Solution. L.H.L. $= \lim_{x \to \frac{5}{2}^-} [x] = \lim_{h \to 0} \left[\frac{5}{2} - h\right] = 2$ $\left[\textit{Putting } x = \frac{5}{2} - h \right]$

$$\text{R.H.L.} = \lim_{x \to \frac{5}{2}^+} [x] = \lim_{h \to 0} \left[\frac{5}{2} + h\right] = 2 \qquad \left[\text{Putting } x = \frac{5}{2} + h\right]$$

$$\therefore \quad \lim_{x \to \frac{5}{2}} [x] = \mathbf{2}.$$

EXAMPLE 6. *Find* $\lim_{x \to 0} f(x)$ *and* $\lim_{x \to 1} f(x)$, *where* $f(x) = \begin{cases} 2x + 3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$.

Solution. *For* $\lim_{x \to 0} f(x)$:

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x + 3)$$

$$= \lim_{h \to 0} [2(0 - h) + 3] = 3 \qquad [\text{Putting } x = 0 - h]$$

$$\text{R.H.L.} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 3(x + 1) = 3 \lim_{x \to 0^+} (x + 1)$$

$$= 3. \lim_{h \to 0} [(0 + h) + 1] = 3 \times 1 = 3 \qquad [\text{Putting } x = 0 + h]$$

$\therefore$ L.H.L. = R.H.L. Hence, $\lim_{x \to 0} f(x)$ exists and is equal to **3**.

For $\lim_{x \to 1} f(x)$:

$$\text{L.H.L.} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3(x + 1) = 3 \lim_{x \to 1^-} (x + 1)$$

$$= 3 \lim_{h \to 0} (1 - h + 1) \qquad [\text{Putting } x = 1 - h]$$

$$= 3 \times (1 + 1) = 3 \times 2 = 6$$

$$\text{R.H.L.} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3(x+1) = 3 \lim_{x \to 1^+} (x+1)$$

$$= 3 \lim_{h \to 0} (1 + h + 1) \qquad [\text{Putting } x = 1 + h]$$

$$= 3 \times (1 + 1) = 3 \times 2 = 6$$

$\therefore$ L.H.L. = R.H.L. = 6. Hence, $\lim_{x \to 1} f(x)$ exists and is equal to **6**.

EXAMPLE 7. *For what value of k does* $\lim_{x \to 1} f(x)$ *exist, where f is defined by the rules*

$$f(x) = \begin{cases} 2kx + 3 & \text{if } x < 1 \\ 1 - kx^2 & \text{if } x > 1 \end{cases}.$$

Solution. *When* $x < 1$, $f(x) = 2kx + 3$

$$\therefore \quad \text{L.H.L.} = \lim_{x \to 1^-} 2kx + 3 = \lim_{h \to 0} 2k(1 - h) + 3 = 2k + 3 \qquad [\text{Putting } x = 1 - h]$$

When $x > 1$, $f(x) = 1 - kx^2$

$\therefore$ R.H.L. $= \lim\limits_{x \to 1^+} 1 - kx^2 = \lim\limits_{h \to 0} 1 - k(1+h)^2 = 1 - k$ [*Putting* $x = 1 + h$]

Now the limits exists if R.H.L. = L.H.L.

i.e., if $2k + 3 = 1 - k \Rightarrow 3k = -2 \Rightarrow k = -\frac{2}{3}$.

EXAMPLE 8. *Let* $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$ *and if* $\lim\limits_{x \to 1} f(x) = f(1)$, *what are the possible values of a and b.*

Solution. R.H.L. $= \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (b - ax)$ [$\because f(x) = b - ax$, *for* $x > 1$]

$= \lim\limits_{h \to 0} [b - a(1+h)]$ [*Putting* $x = 1 + h$, *so that as* $x \to 1$, $h \to 0$]

$= b - a$

L.H.L. $= \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (a + bx)$ [$\because f(x) = a + bx$, *for* $x < 1$]

$= \lim\limits_{h \to 0} [a + b(1-h)]$ [*Putting* $x = 1 - h$, *so that as* $x \to 1$, $h \to 0$]

$= a + b$

It is given that $\lim\limits_{x \to 1} f(x) = f(1)$

$\therefore$ R.H.L. $= f(1) =$ L.H.L. $\Rightarrow b - a = 4 = a + b$ [$\because f(1) = 4$]

Thus we have the equations

$$a + b = 4 \quad ...(1)$$

$$-a + b = 4 \quad ...(2)$$

Solving (1) and (2), we get $\boldsymbol{a = 0, \ b = 4}$

Hence, the possible values of a and b are 0 and 4 respectively.

EXAMPLE 9. *Let a function f be defined as*

$$f(x) = \begin{cases} x, & \text{if } 0 \le x < \frac{1}{2} \\ 0, & \text{if } x = \frac{1}{2} \\ x - 1, & \text{if } \frac{1}{2} < x \le 1 \end{cases}$$

Establish the existence of $\lim\limits_{x \to \frac{1}{2}} f(x)$.

Solution. R.H.L. $= \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+} (x-1)$ $\left[\because \text{When } x > \frac{1}{2}, f(x) = x - 1\right]$

$= \lim_{h \to 0} \left(\frac{1}{2} + h - 1\right)$ $\left[\text{Putting } x = \frac{1}{2} + h, \text{so that when } x \to \frac{1}{2}, h \to 0\right]$

$= \frac{1}{2} + 0 - 1 = -\frac{1}{2}$

L.H.L. $= \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} x$ $\left[\because \text{When } x < \frac{1}{2}, f(x) = x\right]$

$= \lim_{h \to 0} \left(\frac{1}{2} - h\right)$ $\left[\text{Putting } x = \frac{1}{2} - h, \text{ so that when } x \to \frac{1}{2}, h \to 0\right]$

$= \frac{1}{2} - 0 = \frac{1}{2}$

Now, R.H.L. $\neq$ L.H.L. Hence, $\lim_{x \to \frac{1}{2}} f(x)$ does not exist.

EXERCISE 9.4

1. Evaluate the following limits, if they exist :

(i) $\lim_{x \to 0} |x|$ (ii) $\lim_{x \to 2} \frac{|x-2|}{x-2}$ (iii) $\lim_{x \to 1} [x]$

(iv) $\lim_{x \to 2} x - [x]$ (v) $\lim_{x \to 1^+} \frac{[x-1]}{x-1}$ (vi) $\lim_{x \to 0^-} \frac{x}{|x| + x^2}$

2. Evaluate :

(i) $\lim_{x \to 3^+} \frac{x}{[x]}$ (ii) $\lim_{x \to 3^-} \frac{x}{[x]}$ (iii) $\lim_{x \to 0} \frac{3x + |x|}{7x - 5|x|}$

3. Evaluate the left hand limit and right hand limit of the following function at $x = 1$:

$f(x) = \begin{cases} 5x - 4, & \text{if } 0 < x \leq 1 \\ 4x^2 - 3x, & \text{if } 1 < x < 2 \end{cases}$. Does $\lim_{x \to 1} f(x)$ exist.

4. Find k so that $\lim_{x \to 2} f(x)$ exists where $f(x) = \begin{cases} 2x + 3, & \text{if } x \leq 2 \\ x + 3k, & \text{if } x > 2 \end{cases}$.

5. If $f(x) = \begin{cases} x + \frac{1}{2}, & \text{if } x > \frac{1}{2} \\ 0, & \text{if } x = \frac{1}{2} \\ 2x, & \text{if } x < \frac{1}{2} \end{cases}$, find $\lim_{x \to \frac{1}{2}} f(x)$, if it exists.

6. Let $f(x) = \begin{cases} ax + b, & \text{if } x > 1 \\ 3bx - 2a + 1, & \text{if } x < 1 \end{cases}$. Find a relation between a and b so that $\underset{x \to 1}{\text{Lim}} f(x)$ exists.

ANSWERS

1. (i) 0 (ii) does not exist; R.H.L. = 1, L.H.L. = – 1
(iii) does not exist; L.H.L = 0, R.H.L. = 1
(iv) does not exist; L.H.L. = 1, R.H.L. = 0 (v) 0 (vi) – 1
2. (i) 1 (ii) $\frac{3}{2}$ (iii) does not exist
3. L.H.L. = 1, R.H.L. = 1; exists
4. $k = \frac{5}{3}$
5. 1
6. $3a - 2b - 1 = 0$

CONTINUITY

9.7. ▶ CONTINUITY OF A FUNCTION AT A POINT

Generally, we observe that a function is continuous at a fixed point if we can draw the graph of the function around that point without any break in the graph or more precisely without lifting the pen from the plane of the paper.

Geometrically we say that a function $y = f(x)$ is continuous at $x = a$, if the graph of the function $y = f(x)$ is continuous (without any break) at $x = a$.

Definition. *A function $f(x)$ is said to be continuous at a point $x = a$ of its domain if*

(i) *$f(a)$ exists, i.e., $f(a)$ is finite, definite and real.*

(ii) *$\underset{x \to a}{\text{Lim}} f(x)$ exists.*

(iii) *$\underset{x \to a}{\text{Lim}} f(x) = f(a)$*

Thus, *$f(x)$ is continuous at $x = a \Leftrightarrow \underset{x \to a}{\text{Lim}} f(x) = f(a) \Leftrightarrow \underset{x \to a^-}{\text{Lim}} f(x) = \underset{x \to a^+}{\text{Lim}} f(x) = f(a)$.*

In other words, a function f is said to be continuous at $x = a$ if the left hand limit (L.H.L.), the right hand limit (R.H.L.) and the value of the function at $x = a$ exist and are equal to each other.

A function $f(x)$ is said to be left continuous or continuous from the left at $x = a$, iff

(i) $\underset{x \to a^-}{\text{Lim}} f(x)$ exists and (ii) $\underset{x \to a^-}{\text{Lim}} f(x) = f(a)$

A function $f(x)$ is said to be right continuous or continuous from the right at $x = a$, iff

(i) $\underset{x \to a^+}{\text{Lim}} f(x)$ exists and (ii) $\underset{x \to a^+}{\text{Lim}} f(x) = f(a)$

Thus, it follows that a function $f(x)$ is continuous at $x = a$ iff it is both left as well as right continuous at $x = a$.

Remark:

R.H.L. = $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h)$

L.H.L. = $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h)$, where $h \to 0$ through positive values.

Thus we may also define continuity at a point as follows :

A function $f(x)$ is said to be continuous at $x = a$, if

$$\lim_{h \to 0} f(a + h) = \lim_{h \to 0} f(a - h) = f(a), \text{ where } h \to 0 \text{ through positive values.}$$

9.7.1. Discontinuous Function

A function $f(x)$ is said to be discontinuous at $x = a$ if it is not continuous at $x = a$. In other words, a function $f(x)$ fails to be continuous at $x = a$ in either of the following cases :

(i) f is not defined at $x = a$ *i.e.*, $f(a)$ does not exist

(ii) $\lim_{x \to a} f(x)$ does not exist, which happens if either $\lim_{x \to a^-} f(x)$ does not exist or, $\lim_{x \to a^+} f(x)$ does not exist or both $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal.

(iii) $\lim_{x \to a} f(x)$ exists but it is not equal to $f(a)$

The point 'a' where the function $f(x)$ is discontinuous is known as point of discontinuity.

9.8. ▶ ALGEBRA OF CONTINUOUS FUNCTIONS

Theorem. 1 ***If f and g be continuous at a, then***

(i) $f + g$ is continuous at a.

(ii) $f - g$ is continuous at a.

(iii) cf is continuous at a, where c is a real number.

(iv) fg is continuous at a.

(v) $\frac{f}{g}$ is continuous at a, provided $g(a) \neq 0$.

(vi) $\frac{1}{f}$ is continuous at a, provided $f(a) \neq 0$.

Proof. (i) Since f is continuous at $x = a$

$\therefore$ $$\lim_{x \to a} f(x) = f(a) \quad \text{...(i)}$$

Since g is continuous at $x = a$

$\therefore$ $$\lim_{x \to a} g(x) = g(a) \quad \text{...(ii)}$$

Now, $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x) = f(a) + g(a)$ [*Using (i) and (ii)*]

$\therefore f + g$ is continuous at $x = a$.

Proof. (*iv*) Since f and g are continuous at $x = a$.

$$\therefore \quad \lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} g(x) = g(a) \quad \ldots(i)$$

Now,
$$\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x)\, g(x)$$
$$= \lim_{x \to a} f(x) \,.\, \lim_{x \to a} g(x) = f(a) \,.\, g(a) \quad [\textit{Using (i)}]$$
$$= (fg)(a)$$

$\therefore fg$ is continuous at $x = a$.

Proofs of (*ii*), (*iii*), (*v*) and (*vi*) are left as an exercise for the students.

SOLVED EXAMPLES

EXAMPLE 1. *Examine whether the function $f(x) = x^2$ is continuous at $x = 0$.*

Solution. Here we observe that the function $f(x) = x^2$ is defined at the given point *i.e.,* $x = 0$ and its value is 0.

Now,
$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0$$

Thus,
$$\lim_{x \to 0} f(x) = f(0) = 0$$

Hence, $f(x)$ is **continuous** at $x = 0$.

EXAMPLE 2. *If* $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & \text{when } x \neq 1 \\ 2, & \text{when } x = 1 \end{cases}$; *show that $f(x)$ is continuous at $x = 1$.*

Solution.
$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1} (x + 1) = 2$$

Also, $f(1) = 2$ [*Given*]

$$\therefore \quad \lim_{x \to 1} f(x) = f(1) = 2$$

Hence, $f(x)$ is **continuous** at $x = 1$.

EXAMPLE 3. *Discuss the continuity of the function $f(x)$ at $x = 1$, where*

$$f(x) = \begin{cases} \dfrac{3}{2} - x, & \dfrac{1}{2} \leq x < 1 \\ \dfrac{3}{2}, & x = 1 \\ \dfrac{3}{2} + x, & 1 < x \leq 2. \end{cases}$$

Solution. Here $f(x) = \begin{cases} \frac{3}{2} - x, & \frac{1}{2} \le x < 1 \\ \frac{3}{2}, & x = 1 \\ \frac{3}{2} + x, & 1 < x \le 2 \end{cases}$

At $x = 1$: L.H.L. $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \left(\frac{3}{2} - x\right)$

$$= \lim_{h \to 0} \frac{3}{2} - (1 - h) = \frac{1}{2} \qquad \text{[Putting } x = 1 - h\text{]}$$

R.H.L. $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(\frac{3}{2} + x\right)$

$$= \lim_{h \to 0} \frac{3}{2} + (1 + h) = \frac{5}{2} \qquad \text{[Putting } x = 1 + h\text{]}$$

Now, L.H.L. $\neq$ R.H.L. at $x = 1$

$\Rightarrow$ $\lim_{x \to 1} f(x)$ does not exist.

Hence, $f(x)$ is not continuous at $x = 1$.

EXAMPLE 4. *(i) For what value of k is the following function continuous at $x = 2$:*

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ k, & x = 2 \end{cases}$$

(ii) Determine the value of the constant 'k' so that the function

$$f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \ge 0 \end{cases} \text{ is continuous at } x = 0.$$

Solution. *(i)* Here $f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ k, & x = 2 \end{cases}$

$$\therefore \quad \lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

Also, $f(2) = k$ [Given]

As the function is continuous at $x = 2$

$$\therefore \quad \lim_{x \to 2} f(x) = f(2) \Rightarrow \mathbf{k = 4.}$$

(ii) Here $$f(x) = \begin{cases} \dfrac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} \dfrac{kx}{-x}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases} \qquad [\because |x| = -x \text{ for } x < 0]$$

$$\Rightarrow \quad f(x) = \begin{cases} -k, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$$

Now, $f(x)$ will be continuous at $x = 0$ if

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} = f(0)$$

$$\Rightarrow \qquad -k = 3 \Rightarrow k = -3.$$

EXAMPLE 5. *For what value of λ is the function $f(x) = \begin{cases} \lambda(x^2 - 2x), & if\ x \leq 0 \\ 4x + 1, & if\ x > 0 \end{cases}$ continuous at $x = 0$? What about continuity at $x = 1$?*

Solution. **At $x = 0$:** $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \lambda(x^2 - 2x) = \lim\limits_{h \to 0} \lambda(h^2 + 2h) = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4x + 1) = \lim_{h \to 0} (4h + 1) = 4(0) + 1 = 1$$

$\therefore$ $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$ *i.e.*, limit does not exist for any value of λ and hence $f(x)$ is not continuous at $x = 0$ for any value of λ.

At $x = 1$: $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} (4x + 1) = 4(1) + 1 = 5$

Also, $f(1) = 4(1) + 1 = 5$

$\therefore$ $\lim\limits_{x \to 1} f(x) = f(1)$

Hence, $f(x)$ is **continuous** at $x = 1$ for all values of λ.

EXAMPLE 6. *For what value of k, the function $f(x) = \begin{cases} 2x + 1, & x < 2 \\ k, & x = 2 \\ 3x - 1, & x > 2 \end{cases}$ is continuous at $x = 2$.*

Solution. L.H.L. $= \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (2x + 1)$

Putting $x = 2 - h$, we have

$$\text{L.H.L.} = \lim_{h \to 0} f(2 - h) = \lim_{h \to 0} \{2(2 - h) + 1\} = \lim_{h \to 0} (4 - 2h + 1) = 5$$

$$\text{R.H.L.} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 1)$$

Putting $x = 2 + h$, we have

$$\text{R.H.L.} = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} \{3(2+h) - 1\} = \lim_{h \to 0} (6 + 3h - 1) = 5$$

$\therefore$ $\quad$ L.H.L. = R.H.L. = 5

The function will be continuous at $x = 2$ if $f(2)$ = R.H.L. = L.H.L. *i.e.*, if $\boldsymbol{k = 5}$.

EXAMPLE 7. *Find the relationship between a and b so that the function defined by*

$$f(x) = \begin{cases} ax + 1, \text{ if } & x \le 3 \\ bx + 3, \text{ if } & x > 3 \end{cases} \text{ is continuous at } x = 3.$$

Solution. At $x = 3$:

$$\text{L.H.L.} = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax + 1) = \lim_{h \to 0} [a(3 - h) + 1]$$

[*Putting* $x = 3 - h$]

$$= \lim_{h \to 0} (3a - ah + 1) = 3a + 1$$

$$\text{R.H.L.} = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (bx + 3) = \lim_{h \to 0} [b(3 + h) + 3]$$

[*Putting* $x = 3 + h$]

$$= \lim_{h \to 0} (3b + bh + 3) = 3b + 3$$

Also, $\quad f(3) = a(3) + 1 = 3a + 1$

Now, $f(x)$ will be continuous at $x = 3$ if $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$

i.e., $\quad 3a + 1 = 3b + 3 \;\Rightarrow\; 3(a - b) = 2 \;\Rightarrow\; a - b = \dfrac{2}{3} \;\Rightarrow\; a = b + \dfrac{2}{3}$

which is the required relationship between a and b from which we can find the value of a for any arbitrary value of b.

EXAMPLE 8. *If the function* $f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$ *is continuous at $x = 1$, find the values of a and b.*

Solution. At $x = 1$:

$$\text{R.H.L.} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3ax + b) = \lim_{h \to 0} [3a(1 + h) + b]$$

[*Putting* $x = 1 + h$]

$$= \lim_{h \to 0} (3a + 3ah + b) = 3a + b$$

$$\therefore \quad \text{L.H.L.} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (5ax - 2b) = \lim_{h \to 0} [5a(1 - h) - 2b]$$

[*Putting* $x = 1 - h$]

$$= \lim_{h \to 0} (5a - 5ah - 2b) = 5a - 2b$$

Also, $f(1) = 11$ [*Given*]

Since the function $f(x)$ is given to be continuous at $x = 1$

$\therefore$ L.H.L. = R.H.L. = $f(1)$

i.e., $3a + b = 11$ ...(1)

and $5a - 2b = 11$ ...(2)

Solving (1) and (2), we get $\boldsymbol{a = 3, \ b = 2}$.

EXAMPLE 9. *Discuss the continuity of the function $f(x) = |x|$ at $x = 0$.*

Solution. We have $f(x) = |x|$

$$\therefore \quad f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}.$$

Clearly, $f(x)$ is defined at $x = 0$ and $f(0) = 0$.

$$\text{L.H.L.} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = \lim_{h \to 0} (-(0 - h)) = 0 \qquad [\textit{Putting } x = 0 - h]$$

$$\text{R.H.L.} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x) = \lim_{h \to 0} (0 + h) = 0 \qquad [\textit{Putting } x = 0 + h]$$

$$\therefore \quad \text{L.H.L.} = \text{R.H.L.} \quad \Rightarrow \quad \lim_{x \to 0} f(x) = 0$$

$\therefore$ $\lim_{x \to 0} f(x) = f(0) = 0$. Hence, $f(x)$ is **continuous** at $x = 0$.

EXAMPLE 10. *If $f(x) = \begin{cases} \dfrac{|x-2|}{2-x}, & x \neq 2 \\ -1, & x = 2 \end{cases}$; find whether or not, f is continuous at $x = 2$.*

Solution. Here $f(x) = \begin{cases} \dfrac{|x-2|}{2-x}, & x \neq 2 \\ -1, & x = 2 \end{cases}$

$$\Rightarrow \quad f(x) = \begin{cases} -\left(\dfrac{x-2}{2-x}\right), & \text{if } x < 2 \\ -1, & \text{if } x = 2 \\ \dfrac{x-2}{2-x}, & \text{if } x > 2 \end{cases} \quad \Rightarrow \quad f(x) = \begin{cases} 1, & \text{if } x < 2 \\ -1, & \text{if } x = 2 \\ -1, & \text{if } x > 2 \end{cases}$$

At $\boldsymbol{x = 2}$: $\quad$ L.H.L. $= \lim_{x \to 2^-} 1 = 1$

R.H.L. $= \lim_{x \to 2^+} (-1) = -1$

$\therefore$ L.H.L. $\neq$ R.H.L. $\Rightarrow$ $\lim_{x \to 2} f(x)$ does not exist.

Hence, $f(x)$ is **discontinuous** at $x = 2$.

EXAMPLE 11. *If* $f(x) = \begin{cases} \dfrac{x-5}{|x-5|} + a, & if \ x < 5 \\ a + b, & if \ x = 5 \\ \dfrac{x-5}{|x-5|} + b, & if \ x > 5 \end{cases}$

is a continuous function, then find a, b.

Solution. At $x = 5$:

$$\text{L.H.L.} = \lim_{x \to 5-} f(x) = \lim_{x \to 5-} \left(\frac{x-5}{|x-5|} + a \right)$$

$$= \lim_{h \to 0} \left(\frac{(5-h)-5}{|(5-h)-5|} + a \right) \quad \text{[Putting } x = 5-h]$$

$$= \lim_{h \to 0} \left(\frac{-h}{|-h|} + a \right) = \lim_{h \to 0} \left(-\frac{h}{h} + a \right) = -1 + a$$

$$\text{R.H.L.} = \lim_{x \to 5+} f(x) = \lim_{x \to 5+} \left(\frac{x-5}{|x-5|} + b \right)$$

$$= \lim_{h \to 0} \left(\frac{5+h-5}{|5+h-5|} + b \right) \quad \text{[Putting } x = 5+h]$$

$$= \lim_{h \to 0} \left(\frac{h}{|h|} + b \right) = \lim_{h \to 0} \left(\frac{h}{h} + b \right) = 1 + b$$

Also, $f(5) = a + b$

Now, $f(x)$ will be continuous at $x = 5$ if L.H.L. = R.H.L. = $f(5)$

i.e., $a - 1 = b + 1 = a + b$

$\Rightarrow$ $a - 1 = a + b$ and $b + 1 = a + b$

$\Rightarrow$ $b = -1$ and $a = 1$

Hence, the function will be continuous at $x = 5$ when $a = 1$, $b = -1$.

EXERCISE 9.5

1. Check the continuity of the following functions at the indicated points :

   (*i*) $f(x) = 2x + 3$ at $x = 1$

   (*ii*) $f(x) = 5x - 3$ at $x = -3$ and $x = 5$

2. Examine the continuity at $x = 2$ if $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{when } x \neq 2 \\ 0, & \text{when } x = 2 \end{cases}$

3. Discuss the continuity at $x = 3$ if $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x^2 - 2x - 3}, & \text{if } x \neq 3 \\ \dfrac{5}{3}, & \text{if } x = 3 \end{cases}$.

4. Which of the following functions are continuous at the indicated points :

(i) $f(x) = \begin{cases} x, & \text{if } x < 0 \\ x^2, & \text{if } x \geq 0 \end{cases}$ at $x = 0$ (ii) $f(x) = \begin{cases} 2x - 1, & \text{if } x < 0 \\ 2x + 1, & \text{if } x \geq 0 \end{cases}$ at $x = 0$

5. Discuss the continuity of the following functions at $x = 1$:

(i) $f(x) = \begin{cases} 5x - 4, & 0 < x < 1 \\ 4x^2 - 3x, & 1 \leq x < 2 \end{cases}$ (ii) $f(x) = \begin{cases} x^2 + 2, & \text{if } x > 1 \\ 3, & \text{if } x < 1 \\ 2x + 1, & \text{if } x = 1 \end{cases}$

6. Discuss the continuity of the function $f(x)$ at $x = \dfrac{1}{2}$:

$$f(x) = \begin{cases} \dfrac{1}{2} - x, & 0 \leq x < \dfrac{1}{2} \\ 1, & x = \dfrac{1}{2} \\ \dfrac{3}{2} - x, & \dfrac{1}{2} < x \leq 1 \end{cases}$$

7. For what value of k is the function $f(x) = \begin{cases} kx^2, & x \geq 1 \\ 4, & x < 1 \end{cases}$ continuous at $x = 1$.

8. For what value of k is the function defined by $f(x) = \begin{cases} k(x^2 + 2), & \text{if } x \leq 0 \\ 3x + 1, & \text{if } x > 0 \end{cases}$ continuous at $x = 0$? Also write whether the function is continuous at $x = 1$.

9. For what value of k is the following function continuous at $x = 2$:

$$f(x) = \begin{cases} \dfrac{\sqrt{5x+2} - \sqrt{4x+4}}{x - 2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$$

10. Find the value of k in each of the following so that the given function is continuous at the indicated point.

(i) $f(x) = \begin{cases} 2x^2 + k, & \text{if } x \geq 0 \\ -2x^2 + k, & \text{if } x < 0 \end{cases}$ at $x = 0$ (ii) $f(x) = \begin{cases} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x}, & -1 \leq x < 0 \\ \dfrac{2x+1}{x-2}, & 0 \leq x \leq 1 \end{cases}$ at $x = 0$

11. Discuss the continuity of the following function at $x = 1$:

$$f(x) = \begin{cases} \dfrac{|x^2 - 1|}{x - 1} & \text{if} \quad x \neq 1 \\ 2 & \text{if} \quad x = 1 \end{cases}.$$

ANSWERS

1. (i) Continuous (ii) Continuous at $x = -3$ and $x = 5$
2. Discontinuous (ii) Discontinuous
3. Discontinuous
4. (i) Continuous
5. (i) Continuous (ii) Continuous
6. Discontinuous
7. $k = 4$
8. $k = \frac{1}{2}$; Continuous at $x = 1$
9. $k = \frac{1}{4\sqrt{3}}$
10. (i) All real values (ii) $k = -\frac{1}{2}$
11. Discontinuous

9.9. ▶ CONTINUITY ON AN INTERVAL

Continuity of a function on an open interval.

A function $f(x)$ is said to be continuous on an open interval (a, b), if it is continuous at every point on (a, b).

Continuity of a function on a closed interval.

A function $f(x)$ is said to be continuous on the closed interval $[a, b]$, if it is continuous at every value of x lying between a and b, continuous to the right of a and to the left of $x = b$

i.e.,
$$\lim_{x \to a+0} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b-0} f(x) = f(b).$$

Remark : A function $f(x)$ is said to be discontinuous on an interval if it is discontinuous even at a single point in an interval.

9.10. ▶ SOME IMPORTANT RESULTS ON CONTINUOUS FUNCTIONS

1. *If f and g are two continuous functions on their common domain D, then*

(i) $f + g$ *is continuous on D*

(ii) $f - g$ *is continuous on D*

(iii) cf *is continuous on D, where c is any real number*

(iv) fg *is continuous on D*

(v) $\frac{f}{g}$ *is continuous on D, provided* $g(x) \neq 0$.

(vi) $\frac{1}{f}$ *is continuous on D, provided* $f(x) \neq 0$.

2. *The constant function is continuous.*
3. *The identity function is continuous.*
4. *Every polynomial function is continuous.*
5. *Every rational function is continuous in its domain.*
6. *The modulus function $|x|$ is continuous.*
7. *The exponential function e^x is continuous.*

SOLVED EXAMPLES

EXAMPLE 1. *Discuss the continuity of the function $f(x) = x^3 + x^2 - 1$.*

Solution. Here $f(x) = x^3 + x^2 - 1$, which is a polynomial function.

Clearly, f is defined at every real number c and $f(c) = c^3 + c^2 - 1$.

Also, $$\lim_{x \to c} f(x) = \lim_{x \to c} (x^3 + x^2 - 1) = c^3 + c^2 - 1$$

Thus, $\lim_{x \to c} f(x) = f(c)$ and hence f is continuous for every real number.

Hence, f is a **continuous** function.

EXAMPLE 2. *Discuss the continuity of the function f defined by $f(x) = \frac{1}{x}$, $x \neq 0$.*

Solution. Take any non-zero real number c.

Clearly, f is defined at every real number c, $c \neq 0$ and $f(c) = \frac{1}{c}$

Also, $$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{1}{x} = \frac{1}{c}$$

Thus, $\lim_{x \to c} f(x) = f(c)$ and hence f is continuous at every point in the domain of f.

EXAMPLE 3. *Locate the point of discontinuity (if any) for the function*

$$f(x) = \begin{cases} x^3 - x^2 + 2x - 2, & \text{if } x \neq 1 \\ 4, & \text{if } x = 1 \end{cases}.$$

Solution. Here $$f(x) = \begin{cases} x^3 - x^2 + 2x - 2, & \text{if } x \neq 1 \\ 4, & \text{if } x = 1 \end{cases}$$

When $x > 1$ and $x < 1$, $f(x) = x^3 - x^2 + 2x - 2$, which being a polynomial, is continuous.

$\therefore$ $f(x)$ is continuous for $x < 1$ and $x > 1$.

We now consider the point $x = 1$.

At $x = 1$: L.H.L. $= \lim_{x \to 1^-} (x^3 - x^2 + 2x - 2)$

$$= \lim_{h \to 0} [(1-h)^3 - (1-h)^2 + 2(1-h) - 2] = 0 \quad [\textit{Putting } x = 1-h]$$

R.H.L. $= \lim_{x \to 1^+} (x^3 - x^2 + 2x - 2)$

$$= \lim_{h \to 0} [(1+h)^3 - (1+h)^2 + 2(1+h) - 2] = 0 \quad [\textit{Putting } x = 1+h]$$

$$\therefore \quad \lim_{x \to 1} f(x) = 0 \quad [\because \text{L.H.L.} = \text{R.H.L.}]$$

Also, $f(1) = 4$

$$\therefore \quad \lim_{x \to 1} f(x) \neq f(1)$$

Hence, f is discontinuous at $x = 1$. At all other points, f is continuous.

EXAMPLE 4. *Find all the points of discontinuity of the function f defined by*

$$f(x) = \begin{cases} x+2, & \text{if } x < 1 \\ 0, & \text{if } x = 1 \\ x-2, & \text{if } x > 1 \end{cases}$$

Solution. The given function is defined for all real numbers.

When $x < 1$, $f(x) = x + 2$, which being a polynomial is continuous for all $x < 1$.

When $x > 1$, $f(x) = x - 2$, which being a polynomial is continuous for all $x > 1$.

We now consider the point $x = 1$.

We have $$f(x) = \begin{cases} x+2, & \text{if } x < 1 \\ 0, & \text{if } x = 1 \\ x-2, & \text{if } x > 1 \end{cases}$$

At $x = 1$: L.H.L. $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+2) = \lim_{h \to 0} ((1-h)+2) = 3$ [*Putting* $x = 1-h$]

R.H.L. $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-2) = \lim_{h \to 0} ((1+h)-2) = -1$ [*Putting* $x = 1+h$]

Now, L.H.L. $\neq$ R.H.L.

Thus $f(x)$ is not continuous at $x = 1$.

Hence $x = 1$ is the only point of discontinuity of f.

EXAMPLE 5. *If a function f is defined as* $f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases}$.

Show that f is continuous everywhere except at $x = 4$.

Solution. Here $$f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0, & x = 4 \end{cases}$$

Now, $$|x-4| = \begin{cases} -(x-4), & x < 4 \\ x-4, & x \ge 4 \end{cases}$$

$$\therefore \quad f(x) = \begin{cases} \dfrac{-(x-4)}{x-4} = -1, & \text{if } x < 4 \\ \dfrac{x-4}{x-4} = 1, & \text{if } x > 4 \\ 0, & \text{if } x = 4 \end{cases}$$

When $x < 4$, $f(x) = -1$, which being a constant function, is continuous for all $x < 4$.

When $x > 4$, $f(x) = 1$, which being a constant function, is continuous for all $x > 4$.

We now consider the point $x = 4$.

At $x = 4$: L.H.L. $= \lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} (-1) = -1$

R.H.L. $= \lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^+} (1) = 1$

$\therefore$ L.H.L. $\neq$ R.H.L. $\Rightarrow \lim\limits_{x \to 4} f(x)$ does not exist.

$\therefore f(x)$ is not continuous at $x = 4$.

Hence, $f(x)$ is continuous everywhere except at $x = 4$.

EXAMPLE 6. *In a function* $p(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax+b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$, *determine the values of constants so that the given functions are continuous :*

Solution. We have $$p(x) = \begin{cases} 5, & \text{if } x \le 2 \\ ax+b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \ge 10 \end{cases}$$

When $x < 2$, $p(x) = 5$, which being a constant function, is continuous for all $x < 2$.

When $2 < x < 10$, $p(x) = ax + b$, which being a polynomial, is continuous for all $x \in (2, 10)$

When $x > 10$, $p(x) = 21$, which being a constant function, is continuous for all $x > 10$.

We now consider the continuity of $p(x)$ at $x = 2$ and $x = 10$.

At $x = 2$: L.H.L $= \lim\limits_{x \to 2^-} (5) = 5$

R.H.L $= \lim\limits_{x \to 2^+} (ax+b) = \lim\limits_{h \to 0} [a(2+h)+b] = 2a+b$ [*Putting* $x = 2 + h$]

and $p(2) = 5$

$\therefore$ For continuity at $x = 2$; $2a + b = 5$ ...(1)

At $x = 10$: L.H.L. $= \lim\limits_{x \to 10^-} (ax+b) = \lim\limits_{h \to 0} [a(10-h)+b] = 10a + b$ [*Putting* $x = 10 - h$]

$$\text{R.H.L.} = \lim_{x \to 10^+} (21) = 21$$

and $$p(10) = 21$$

$\therefore$ For continuity at $x = 10$; $10a + b = 21$ ...(2)

Solving (1) and (2), we get $a = \mathbf{2}, b = \mathbf{1}$.

EXAMPLE 7. *Find all the points of discontinuity of the function f defined by*

$$f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3. \\ 6x + 2, & \text{if } x \ge 3 \end{cases}$$

Solution. Here $f(x) = \begin{cases} |x| + 3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \ge 3 \end{cases}$

When $x < -3$, $f(x) = |x| + 3 = -x + 3$, which being a polynomial, is continuous for all $x < -3$.
$[\because |x| = -x \text{ if } x < 0]$

When $-3 < x < 3$, $f(x) = -2x$, which being a polynomial, is continuous for all x lying between -3 and 3 *i.e.*, $x \in (-3, 3)$

When $x > 3$, $f(x) = 6x + 2$, which being a polynomial, is continuous for all $x > 3$.

We now examine the continuity of $f(x)$ at $x = -3$ and $x = 3$.

At $x = -3$: $\text{L.H.L.} = \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} |x| + 3 = -x + 3$

$$= \lim_{h \to 0} \{-(-3 - h) + 3\} = 3 + 3 = 6 \qquad [\textit{Putting } x = -3 - h]$$

$$\text{R.H.L.} = \lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (-2x)$$

$$= \lim_{h \to 0} \{-2(-3 + h)\} = 6 \qquad [\textit{Putting } x = -3 + h]$$

Also, $f(-3) = |-3| + 3 = 3 + 3 = 6$

$\therefore$ L.H.L. = R.H.L. $= f(-3)$. Thus, $f(x)$ is **continuous** at $x = -3$.

At $x = 3$: $\text{L.H.L.} = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-2x)$

$$= \lim_{h \to 0} \{-2(3 - h)\} = -6 \qquad [\textit{Putting } x = 3 - h]$$

$$\text{R.H.L.} = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + 2)$$

$$= \lim_{h \to 0} \{6(3 + h) + 2\} = 20 \qquad [\textit{Putting } x = 3 + h]$$

$\therefore$ L.H.L. $\neq$ R.H.L. $\Rightarrow \lim_{x \to 3} f(x)$ does not exist.

Thus, $f(x)$ is **discontinuous** at $x = 3$.

Hence, $f(x)$ is continuous at all points except $x = 3$ *i.e.*, $x = 3$ is the only point of discontinuity.

EXERCISE 9.6

1. In the following, determine the value of the constant so that the given function is continuous.

$$f(x)=\begin{cases} kx+5 & \text{if } x\le 2 \\ x-1 & \text{if } x>2 \end{cases}$$

2. Discuss the continuity of the function $f(x)=\begin{cases} x^2 & \text{if } x>0 \\ x & \text{if } x\le 0 \end{cases}$.

3. Locate the point of discontinuity (if any) for the following functions :

(i) $f(x)=\begin{cases} 2x-1 & \text{if } x<2 \\ \dfrac{3x}{2} & \text{if } x\ge 2 \end{cases}$ (ii) $f(x)=\begin{cases} \dfrac{x}{|x|} & \text{if } x\ne 0 \\ 0 & \text{if } x=0 \end{cases}$ (iii) $f(x)=\begin{cases} \dfrac{x^4-16}{x-2} & \text{if } x\ne 2 \\ 16 & \text{if } x=2 \end{cases}$

4. Discuss the continuity of the function f, where f is defined by :

(i) $f(x)=\begin{cases} 2x, & \text{if } x<0 \\ 0, & \text{if } 0\le x\le 1 \\ 4x, & \text{if } x>1 \end{cases}$ (ii) $f(x)=\begin{cases} -2, & \text{if } x\le -1 \\ 2x, & \text{if } -1<x\le 1 \\ 2, & \text{if } x>1 \end{cases}$

5. Find the values of a and b such that the functions defined below are continuous :

(i) $f(x)=\begin{cases} x+2 \text{ , if } & x\le 2 \\ ax+b \text{ , if } & 2<x<5 \\ 3x-2 \text{ , if } & x\ge 5 \end{cases}$ (ii) $f(x)=\begin{cases} ax^2+b \text{ , } & x<2 \\ 2 \text{ , } & x=2 \\ 2ax-b \text{ , } & x>2 \end{cases}$

ANSWERS

1. $k=-2$
2. Continuous for all $x\in \mathbf{R}$
3. (i) Continuous for all $x\in \mathbf{R}$ (ii) Discontinuous at $x=0$ (iii) Discontinuous at $x=2$
4. (i) Continuous for all real x except $x=1$ (ii) Continuous for all $x\in \mathbf{R}$
5. (i) $a=3,\ b=-2$ (ii) $a=\frac{1}{2}, b=0$

MULTIPLE CHOICE QUESTIONS & ASSERTION-REASON BASED QUESTIONS

Type I. Multiple Choice Questions :

1. $\lim\limits_{x\to 1}\dfrac{x^m-1}{x^n-1}$ is equal to

(a) 1 (b) $\dfrac{m}{n}$ (c) $-\dfrac{m}{n}$ (d) $\dfrac{m^2}{n^2}$

2. $\lim\limits_{x\to 0}\dfrac{(1+x)^n-1}{x}$ is equal to

(a) n (b) 1 (c) $-n$ (d) 0

3. $\lim\limits_{x \to \infty} \dfrac{1+2+3+....+n}{n^2}$, $n \in N$ is equal to

(a) 0 (b) 1 (c) $\dfrac{1}{2}$ (d) $\dfrac{1}{4}$

4. $\lim\limits_{x \to 1} [x-1]$, where [·] is a greatest integer function, is equal to

(a) 1 (b) 2 (c) 0 (d) does not exist

5. $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ is equal to

(a) 1 (b) -1 (c) 0 (d) does not exist

6. $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x-a}$, $n \in N$ is equal to

(a) $(n-1)a^{n-1}$ (b) $(n-1)a^{n-2}$ (c) na^{n-1} (d) none of these.

7. $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x}-1}{x}$ is equal to

(a) -1 (b) 0 (c) 2 (d) $\dfrac{1}{2}$

8. The value of $\lim\limits_{x \to 3} \dfrac{x^2-9}{x-3}$ is

(a) 3 (b) 9 (c) -3 (d) 6

9. The value of $\lim\limits_{x \to 2} \dfrac{x^5-32}{x-2}$ is

(a) 20 (b) 40 (c) 80 (d) 100

10. The value of $\lim\limits_{x \to 0} \dfrac{a^x-1}{b^x-1}$ is

(a) 0 (b) $\dfrac{\log a}{\log b}$ (c) $\log \dfrac{a}{b}$ (d) does not exist

11. The value of $\lim\limits_{x \to 1} \dfrac{(\sqrt{x}-1)(2x-3)}{2x^2+x-3}$ is

(a) $-\dfrac{1}{5}$ (b) $-\dfrac{3}{2}$ (c) $\dfrac{1}{5}$ (d) $-\dfrac{1}{10}$

12. The value of $\lim\limits_{x \to 0^-} [x]$ is

(a) -1 (b) 0 (c) 1 (d) does not exist

13. The function $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$ is not continuous at

(a) $x = 0$ (b) $x = 1$ (c) $x = -1$ (d) none of these.

14. The value of λ for which $f(x)=\begin{cases}\lambda x^2+8x, & x\le 2\\ 2x+6, & x>2\end{cases}$ is continuous at $x=2$ is

(a) 0 (b) 2 (c) $\frac{1}{2}$ (d) $-\frac{3}{2}$

15. The point(s), at which the function f given by $f(x)=\begin{cases}\dfrac{x}{|x|}, & x<0\\ -1, & x\ge 0\end{cases}$ is continuous, is/are :

(a) $x\in R$ (b) $x=0$ (c) $x\in R-\{0\}$ (d) $x=-1$ and 1

16. The number of points where $f(x)=|x-1|+|x+3|$ is not differentiable are

(a) 0 (b) 1 (c) 2 (d) infinite

17. $f(x)=\dfrac{x^2-9}{x-3}$ is not defined at $x=3$. What value should be assigned to $f(3)$ for continuity of $f(x)$ at $x=3$?

(a) 3 (b) 0 (c) 6 (d) 9

18. The number of points of discontinuity of f defined by $f(x)=|x|-|x+1|$ is

(a) 1 (b) -1

(c) R (d) No point of discontinuity

19. The value of k, so that the function defined by $f(x)=\begin{cases}kx+1, & \text{when } x\le 5\\ 3x-5, & \text{when } x>5\end{cases}$ is continuous at $x=5$ is

(a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{9}{5}$ (d) 5

20. The points of discontinuity for the function $f(x)=[x]$, where $x\in[-2,2)$ is

(a) $-2, 0, 1$ (b) $-1, 0, 1$ (c) $-2, 2$ (d) 0, 1

21. The points at which the function $f(x)=\dfrac{1}{\log|x|}$ is discontinuous is

(a) $-2, 0, 1$ (b) $-1, 0, 1$ (c) $-2, 2$ (d) 0, 1

Type II - Assertion-Reasoning Questions :

In the following questions a statement of assertion (A) is followed by a statement of reason (R). Choose the correct option from the given options.

(a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A).

(b) Both Assertion (A) and Reason (R) are true but Reason (R) is not the correct explanation of Assertion (A).

(c) Assertion (A) is true but Reason (R) is false.

(d) Assertion (A) is false but Reason (R) is true.

22. Assertion (A) : $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{2x} = \dfrac{1}{2}$

Reason (R) : The limit of the quotient of two functions is equal to the quotients of their limits

i.e., $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, provided $\lim\limits_{x \to a} g(x) \neq 0$.

23. Assertion (A) : $\lim\limits_{x \to 0} \dfrac{(x+2)^3 - 8}{x} = 12$

Reason (R) : $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = n\,a^{n-1}$

24. Assertion (A) : $\lim\limits_{x \to 0} \dfrac{a^x - 1}{b^x - 1} = \dfrac{\log a}{\log b}$

Reason (R) : $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = 1$

25. Assertion (A) : If $f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases}$, then $\lim\limits_{x \to 1} f(x)$ does not exist.

Reason (R) : R.H.L. = – 2 and L.H.L. = 0.

26. Assertion (A) : $\lim\limits_{x \to 2} \dfrac{|x-2|}{x-2}$ does not exist.

Reason (R) : The limit of a function exist if R.H.L = L.H.L = 0.

27. Assertion : $f(x) = |x|$ is continuous at $x = 0$.

Reason : The left hand limit and right hand limit of $f(x)$ at $x = 0$ are equal.

28. Assertion : The value of k for which the function $f(x) = \begin{cases} 2x - 1, & x < 2 \\ k, & x = 2 \\ x + 1, & x > 2 \end{cases}$ is continuous at $x = 2$ is 4.

Reason : A function $f(x)$ is continuous at a point $x = a$ of its domain if $\lim\limits_{x \to a} f(x) = f(a)$

29. Assertion : The greatest integer function $f(x) = [x]$ may be continuous.

Reason : $f(x)$ is continuous at all real values of x except all integral points.

ANSWERS

Type I. Multiple Choice Questions :

1. (*b*)	**2.** (*a*)	**3.** (*c*)	**4.** (*d*)	**5.** (*d*)	**6.** (*c*)
7. (*d*)	**8.** (*d*)	**9.** (*c*)	**10.** (*b*)	**11.** (*d*)	**12.** (*a*)
13. (*a*)	**14.** (*d*)	**15.** (*a*)	**16.** (*c*)	**17.** (*c*)	**18.** (*d*)
19. (*c*)	**20.** (*b*)	**21.** (*b*)			

Type II - Assertion-Reasoning Questions :

22. (*b*)	**23.** (*a*)	**24.** (*c*)	**25.** (*a*)	**26.** (*c*)	**27.** (*b*)
28. (*d*)	**29.** (*a*)				