# 11 Differentiation

## DERIVATIVE AT A GIVEN POINT

The concept of the derivative of a real valued function at a point is intimately connected with that of the slope of the tangent to the curve (graph of the function) at the given point on it.

Let $y = f(x)$ be a function which is defined in some $\delta$-neighbourhood of $c$ where $\delta$ is a small positive real number and $f$ is continuous at $x = c$ *i.e.* at the point $P(c, f(c))$. Let $Q\ (c + h, f(c + h))$ be *any other* point on the graph of $f$ (shown in fig. 11.1) where $c + h$ lies in the neighbourhood of $c$ *i.e.* $h$ is very small, $h \neq 0$, it may be positive or negative. In fig. 11.1, we have shown the case when $h > 0$.

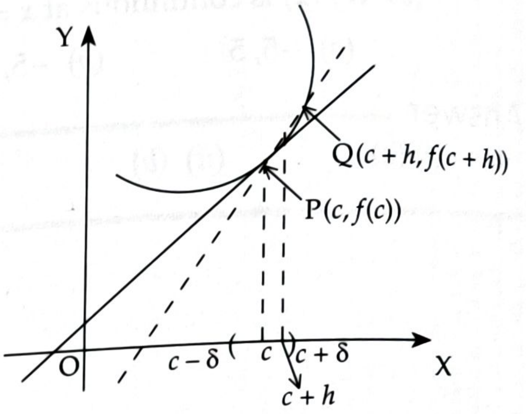


*Fig. 11.1.*

$$\text{Slope of the line PQ} = \frac{f(c+h) - f(c)}{(c+h) - c} = \frac{f(c+h) - f(c)}{h}.$$

Now, if we hold the point P fixed and let Q approach P closer and closer, the value of the slope varies. The line PQ *rotates* about the point P till it reaches its *limiting position* when Q ultimately coincides with P. In this position, the line PQ becomes a tangent to the curve (graph of $f$) at P. The limiting value of the slope when $Q \to P$ *i.e.* $\underset{h \to 0}{\text{Lt}} \dfrac{f(c+h) - f(c)}{h}$ (if it exists) is called the **derivative** of $f$ at the point $x = c$, it is denoted by $f'(c)$ (read as $f$-dash $c$).

The indicated limit may exist at some points and may fail to exist at some other points. Moreover, we can approach P either from left or from right. This leads to the following definitions of the derivative at a given point:

*A function $f$ is said to have a **left derivative** at $x = c$ iff $f$ is defined in some (undeleted) left neighbourhood of $c$ and $\underset{h \to 0^-}{\text{Lt}} \dfrac{f(c+h) - f(c)}{h}$ exists (finitely).*

*The value of this limit is called the **left derivative** at $x = c$ and is denoted by $f'_-(c)$ or $Lf'(c)$ i.e.*

$$f'_-(c) = \underset{h \to 0^-}{\text{Lt}} \frac{f(c+h) - f(c)}{h}.$$

*A function $f$ is said to have a **right derivative** at $x = c$ iff $f$ is defined in some (undeleted) right neighbourhood of $c$ and $\underset{h \to 0^+}{\text{Lt}} \dfrac{f(c+h) - f(c)}{h}$ exists (finitely).*

*The value of this limit is called the* ***right derivative*** *at $x = c$ and is denoted by $f'_+(c)$ or $Rf'(c)$ i.e.*

$$f'_+(c) = \underset{h \to 0^+}{Lt} \frac{f(c+h) - f(c)}{h}.$$

*A function f is said to have a* ***derivative*** *at $x = c$ iff f is defined in some (undeleted) neighbourhood of c and $\underset{h \to 0}{Lt} \frac{f(c+h) - f(c)}{h}$ exists (finitely).*

*The value of this limit is called the* ***derivative*** *at $x = c$ and is denoted by $f'(c)$ i.e.*

$$f'(c) = \underset{h \to 0}{Lt} \frac{f(c+h) - f(c)}{h}.$$

***From the above definitions, it follows that the derivative of a function f at $x = c$ exists iff both the left and right derivatives exist separately at that point and are equal.***

## Geometrically (slope of tangent)

The left derivative (if it exists) at $x = c$ represents the **slope of the tangent** to the curve $y = f(x)$ at the point $P(c, f(c))$ from the left side and the right derivative (if it exists) at $x = c$ represents the slope of the tangent to the curve $y = f(x)$ at the point $P(c, f(c))$ from the right side; and the derivative (if it exists) at $x = c$ represents the slope of the tangent to the curve $y = f(x)$ at the point $P(c, f(c))$. Therefore, the derivative of the function $f$ at $x = c$ will exist iff the tangent to the curve $y = f(x)$ exists on both sides of the point $P(c, f(c))$ in its immediate neighbourhood as well as at $x = c$ itself and the tangent from the left and right coincide and this common tangent is not parallel to $y$-axis.

***Hence, the derivative of a function f at the point $x = c$ will exist iff there is one and only one tangent to the curve $y = f(x)$ at the point $(c, f(c))$ and is not parallel to y-axis.***

*The derivative of a function f at $x = c$ is also called* ***differential coefficient*** *of f at $x = c$.*

*A function f is said to be* ***differentiable*** *at a point c iff the derivative of f at c exists. The process of finding the derivative of a function is called* ***differentiation.***

## Derivative at any point

*A function f is said to have a* ***derivative*** *at any point x iff it is defined in some (undeleted) neighbourhood of the point x and $\underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$ exists (finitely).*

*The value of this limit is called the* ***derivative*** *of f at any point x and is denoted by $f'(x)$ i.e.*

$$f'(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}.$$

## Other notations

There are different notations for the derivative of a function. Sometimes it is denoted by $\frac{d}{dx}(f(x))$ or by $\frac{df}{dx}$.

If the function $f$ is written as $y = f(x)$, then its derivative is written as $\frac{dy}{dx}$ (or $y_1$).

The derivative at $x = c$ is denoted by $\frac{d}{dx}(f(x))/c$ or $\left(\frac{df}{dx}\right)_{x=c}$ or $\left(\frac{dy}{dx}\right)_{x=c}$.

By differentiation from *first principles* or *ab-initio*, we mean differentiation by using definition.

## DERIVATIVE AS INSTANTANEOUS RATE OF CHANGE

Let $y = f(x)$ be a function of $x$ and let independent variable $x$ changes from $x$ to $x + \delta x$ then consequently dependent variable $y$ changes from $y$ to $y + \delta y$.

So, $y + \delta y = f(x + \delta x)$

$$\Rightarrow \quad \delta y = f(x + \delta x) - f(x)$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

$\therefore$ Average rate of change of $y$ per unit change in $x = \dfrac{\delta y}{\delta x}$

As $\delta x \to 0$, the limiting value of $\dfrac{\delta y}{\delta x}$ becomes instantaneous (actual) rate of change of $y$ w.r.t. $x$.

Thus, $\underset{\delta x \to 0}{\text{Lt}} \dfrac{\delta y}{\delta x}$ = Instantaneous rate of change of $y$ w.r.t. $x$

$$\Rightarrow \quad \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x) - f(x)}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{f(x + \delta x) - f(x)}{\delta x} = f'(x) \qquad \left[\because \underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x} = \frac{dy}{dx}\right]$$

***Hence, physically $\dfrac{dy}{dx}$ represents the instantaneous (actual) rate of change of y w.r.t. x.***

## ILLUSTRATIVE EXAMPLES

**Example 1.** *If $f(x) = x^2 - 5x + 7$, find $f'(3)$ and $f'\left(\frac{7}{2}\right)$ from definition. Hence, show that $f'\left(\frac{7}{2}\right) = 2f'(3)$.*

**Solution.** Given $f(x) = x^2 - 5x + 7$.

By def., $f'(3) = \underset{h \to 0}{\text{Lt}} \dfrac{f(3 + h) - f(3)}{h}$

$$= \underset{h \to 0}{\text{Lt}} \frac{[(3 + h)^2 - 5(3 + h) + 7] - [3^2 - 5 \times 3 + 7]}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{h^2 + h}{h} = \underset{h \to 0}{\text{Lt}} (h + 1) = 1.$$

Also $f'\left(\frac{7}{2}\right) = \underset{h \to 0}{\text{Lt}} \dfrac{f\left(\frac{7}{2} + h\right) - f\left(\frac{7}{2}\right)}{h}$ (by def.)

$$= \underset{h \to 0}{\text{Lt}} \frac{\left[\left(\frac{7}{2} + h\right)^2 - 5\left(\frac{7}{2} + h\right) + 7\right] - \left[\left(\frac{7}{2}\right)^2 - 5 \times \frac{7}{2} + 7\right]}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{h^2 + 2h}{h} = \underset{h \to 0}{\text{Lt}} (h + 2) = 2.$$

Now $f'\left(\frac{7}{2}\right) = 2 = 2 \times 1 = 2f'(3)$.

**Example 2.** *Find the derivatives of the following functions from first principles :*

(*i*) $ax + b, a \neq 0, b \neq 0$ (*ii*) $(x-1)(x-2)$ (*iii*) $x^3 - 27$.

**Solution.** (*i*) Let $f(x) = ax + b$.

$$\text{By def., } f'(x) = \underset{h \to 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h} = \underset{h \to 0}{\text{Lt}} \frac{(a(x+h) + b) - (ax+b)}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{ah}{h} = \underset{h \to 0}{\text{Lt}} a = a.$$

(*ii*) Let $f(x) = (x-1)(x-2) = x^2 - 3x + 2$.

$$\text{By def., } f'(x) = \underset{h \to 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{[(x+h)^2 - 3(x+h) + 2] - [x^2 - 3x + 2]}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{h^2 + 2xh - 3h}{h} = \underset{h \to 0}{\text{Lt}} (h + 2x - 3)$$

$$= 0 + 2x - 3 = 2x - 3.$$

(*iii*) Let $f(x) = x^3 - 27$.

$$\text{By def., } f'(x) = \underset{h \to 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h} = \underset{h \to 0}{\text{Lt}} \frac{[(x+h)^3 - 27] - (x^3 - 27)}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{3x^2h + 3xh^2 + h^3}{h} = \underset{h \to 0}{\text{Lt}} (3x^2 + 3xh + h^2)$$

$$= 3x^2 + 3x \times 0 + 0^2 = 3x^2.$$

**Example 3.** *Find the derivatives of the following functions from first principles :*

(*i*) $\frac{1}{x^2}$ (*ii*) $\frac{x^2 - 1}{x}$.

**Solution.** (*i*) Let $f(x) = \frac{1}{x^2}$, note that $f$ is not defined at $x = 0$.

$$\text{By def., } f'(x) = \underset{h \to 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h} = \underset{h \to 0}{\text{Lt}} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{x^2 - (x+h)^2}{h(x+h)^2x^2} = \underset{h \to 0}{\text{Lt}} \frac{-2hx - h^2}{h(x+h)^2x^2}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{-2x - h}{(x+h)^2x^2} = \frac{-2x - 0}{(x+0)^2x^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}, x \neq 0.$$

(*ii*) Let $f(x) = \frac{x^2 - 1}{x} = x - \frac{1}{x}$, note that $f$ is not defined at $x = 0$.

$$\text{By def., } f'(x) = \underset{h \to 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h} = \underset{h \to 0}{\text{Lt}} \frac{\left((x+h) - \frac{1}{x+h}\right) - \left(x - \frac{1}{x}\right)}{h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{h - \left(\frac{1}{x+h} - \frac{1}{x}\right)}{h} = \underset{h \to 0}{\text{Lt}} \frac{h - \frac{x - x - h}{(x+h)x}}{h}$$

$$= \lim_{h\to 0} \frac{h + \frac{h}{(x+h)x}}{h} = \lim_{h\to 0}\left[1 + \frac{1}{(x+h)x}\right]$$

$$= 1 + \frac{1}{(x+0)x} = 1 + \frac{1}{x^2}, \ x \neq 0.$$

**Example 4.** *Find the derivatives of the following functions from first principles :*

(*i*) $\dfrac{2x+3}{x-2}$ (*ii*) $\dfrac{ax+b}{cx+d}$.

**Solution.** (*i*) Let $f(x) = \dfrac{2x+3}{x-2}$, note that $f$ is not defined at $x = 2$.

$$\text{By def., } f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{2(x+h)+3}{(x+h)-2} - \frac{2x+3}{x-2}}{h}$$

$$= \lim_{h\to 0} \frac{(x-2)(2x+3+2h)-(x-2+h)(2x+3)}{h(x+h-2)(x-2)}$$

$$= \lim_{h\to 0} \frac{(x-2)\,2h - h(2x+3)}{h(x+h-2)(x-2)}$$

$$= \lim_{h\to 0} \frac{-7h}{h(x+h-2)(x-2)} = \lim_{h\to 0} \frac{-7}{(x+h-2)(x-2)}$$

$$= \frac{-7}{(x+0-2)(x-2)} = -\frac{7}{(x-2)^2}, \ x \neq 2.$$

(*ii*) Let $f(x) = \dfrac{ax+b}{cx+d}$, note that $f$ is not defined at $x = -\dfrac{d}{c}$.

$$\text{By def., } f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{a(x+h)+b}{c(x+h)+d} - \frac{ax+b}{cx+d}}{h}$$

$$= \lim_{h\to 0} \frac{(ax+b+ah)(cx+d)-(cx+d+ch)(ax+b)}{h(cx+d+ch)(cx+d)}$$

$$= \lim_{h\to 0} \frac{ah(cx+d)-ch(ax+b)}{h(cx+d+ch)(cx+d)} = \lim_{h\to 0} \frac{ad-bc}{(cx+d+ch)(cx+d)}$$

$$= \frac{ad-bc}{(cx+d+0)(cx+d)} = \frac{ad-bc}{(cx+d)^2}, \ x \neq -\frac{d}{c}.$$

**Example 5.** *Find the derivative of* $\sqrt{ax+b}$ *from first principle.*

**Solution.** Let $f(x) = \sqrt{ax+b}$.

$$\text{By def., } f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{a(x+h)+b} - \sqrt{ax+b}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(ax+b)+ah} - \sqrt{ax+b}}{h} \times \frac{\sqrt{(ax+b)+ah} + \sqrt{ax+b}}{\sqrt{(ax+b)+ah} + \sqrt{ax+b}}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{(ax+b)+ah-(ax+b)}{h\left(\sqrt{(ax+b)+ah}+\sqrt{ax+b}\right)}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{ah}{h\left(\sqrt{(ax+b)+ah}+\sqrt{ax+b}\right)}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{a}{\left(\sqrt{(ax+b)+ah}+\sqrt{ax+b}\right)}$$

$$= \frac{a}{\sqrt{ax+b}+\sqrt{ax+b}} = \frac{a}{2\sqrt{ax+b}}.$$

## EXERCISE 11.1

1. Using definition, find the derivatives of the following functions at the indicated points :
   (i) $f(x) = 3x$ at $x = 2$ (ii) $f(x) = x^2 + 2x + 7$ at $x = 3$
   (iii) $f(x) = 1 - 4x^2$ at $x = 1$
2. For the function $f$, given by $f(x) = 2x^3 - 9x^2 + 12x + 9$, show that $f'(1) = f'(2)$.
3. Find the derivatives of the following functions from first principles :

   (i) $3x - 5$ (ii) $(x-3)^2$ (iii) $x^3$ (iv) $(-x)^{-1}$

   (v) $\frac{3}{x^2}$ (vi) $\frac{x+1}{x-1}$ (vii) $x + \frac{1}{x}$ (viii) $\sqrt{3x+4}$

### Answers

1. (i) 3 (ii) 8 (iii) $-8$

3. (i) 3 (ii) $2(x-3)$ (iii) $3x^2$ (iv) $\frac{1}{x^2}$ (v) $-\frac{6}{x^3}$

   (vi) $-\frac{2}{(x-1)^2}$ (vii) $1 - \frac{1}{x^2}$ (viii) $\frac{3}{2\sqrt{3x+4}}$

### Some standard results on derivatives

**1. *The derivative of a constant function is zero.***

**Proof.** Let $f(x) = c$, for all $x \in \mathbf{R}$ where $c$ is a fixed real number.

By def., $f'(x) = \underset{h \to 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h} = \underset{h \to 0}{\text{Lt}} \frac{c - c}{h}$

$= \underset{h \to 0}{\text{Lt}} \frac{0}{h} = \underset{h \to 0}{\text{Lt}}\ 0$ $(\because h \neq 0)$

$= 0.$

*i.e.* $\frac{d}{dx}(c) = 0$, *for all* $x \in R$.

**2. *The derivative of x is 1.***

**Proof.** Let $f(x) = x$

By def., $f'(x) = \underset{h \to 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h} = \underset{h \to 0}{\text{Lt}} \frac{(x+h) - x}{h} = \underset{h \to 0}{\text{Lt}} \frac{h}{h} = \underset{h \to 0}{\text{Lt}}\ 1 = 1$

*i.e.* $\frac{d}{dx}(x) = 1.$

**3. *The derivative of $x^n$ is $nx^{n-1}$, where n is any positive integer.***

**Proof.** Let $f(x) = x^n$, $n \in \mathbf{N}$.

By def., $f'(x) = \underset{h \to 0}{\text{Lt}} \dfrac{f(x+h) - f(x)}{h} = \underset{h \to 0}{\text{Lt}} \dfrac{(x+h)^n - x^n}{h}$

$= \underset{x+h \to x}{\text{Lt}} \dfrac{(x+h)^n - x^n}{(x+h) - x}$ $\quad (\because h \to 0 \Rightarrow x + h \to x)$

$= n . x^{n-1}$ $\quad \left(\text{using } \underset{x \to a}{\text{Lt}} \dfrac{x^n - a^n}{x - a} = na^{n-1}\right)$

*i.e.* $\dfrac{d}{dx}(x^n) = nx^{n-1}$.

This result is known as **power formula.**

## Remark

The above result *i.e.* $\dfrac{d}{dx}(x^n) = nx^{n-1}$ is true for all rational powers of $x$ *i.e.* $n$ can be any rational number. However, we will not prove it here.

**4. *The derivative of $(ax + b)^n$ is $n(ax + b)^{n-1}$ a, where n is any positive integer and a, b are fixed real numbers.***

**Proof.** Let $f(x) = (ax + b)^n$, $n \in \mathbf{N}$.

By def., $f'(x) = \underset{h \to 0}{\text{Lt}} \dfrac{f(x+h) - f(x)}{h} = \underset{h \to 0}{\text{Lt}} \dfrac{(a(x+h) + b)^n - (ax+b)^n}{h}$

$= \underset{h \to 0}{\text{Lt}}\ a . \dfrac{(ax + b + ah)^n - (ax+b)^n}{ah}$

$= a . \underset{ax+b+ah \to ax+b}{\text{Lt}} \dfrac{(ax+b+ah)^n - (ax+b)^n}{(ax+b+ah) - (ax+b)}$ $\quad (\because h \to 0 \Rightarrow ax + b + ah \to ax + b)$

$= a . n(ax + b)^{n-1}$ $\quad \left(\text{using } \underset{x \to a}{\text{Lt}} \dfrac{x^n - a^n}{x - a} = na^{n-1}\right)$

*i.e.* $\dfrac{d}{dx}((ax + b)^n) = n(ax + b)^{n-1}a$.

Moreover, the above result is true for all rational values of $n$.

**5. Algebra of derivatives**

We state some results (without proof) which enable us to evaluate the derivatives in many problems.

***(i) If f is a differentiable function at x and g is the function defined by $g(x) = cf(x)$, then $g'(x) = cf'(x)$, where c is a fixed real number***

*i.e* $\dfrac{d}{dx}(cf(x)) = c . \dfrac{d}{dx}(f(x))$.

*Thus, the derivative of a scalar multiple of the function is the scalar multiple of the derivative of the function.*

***(ii) If f and g are differentiable functions at x and if h is the function defined by $h(x) = f(x) + g(x)$, then $h'(x) = f'(x) + g'(x)$***

*i.e.* $\dfrac{d}{dx}(f(x) + g(x)) = \dfrac{d}{dx}(f(x)) + \dfrac{d}{dx}(g(x))$.

*Thus, the derivative of the sum of two functions is the sum of their derivatives.*

(iii) *If f and g are differentiable functions at x and h is the function defined by* $h(x) = f(x) - g(x)$, *then* $h'(x) = f'(x) - g'(x)$

*i.e.* $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$.

*Thus, the derivative of the difference of two functions is the difference of their derivatives.*

(iv) *If f and g are differentiable functions at x and if h is the function defined by* $h(x) = f(x)\,g(x)$, *then* $h'(x) = f(x)\,g'(x) + g(x)\,f'(x)$

*i.e.* $\frac{d}{dx}(f(x)g(x)) = f(x).\frac{d}{dx}(g(x)) + g(x).\frac{d}{dx}(f(x))$.

*Thus, the derivative of the product of two differentiable functions*
= *first function × derivative of second function + second function × derivative of first function.*

This is known as **product rule** or **Leibnitz's rule.**

**Extension of the product rule**

*If f, g and h are three differentiable functions at x, then*

$$\frac{d}{dx}(f(x)g(x)h(x)) = f(x)\,g(x)\frac{d}{dx}(h(x)) + f(x)h(x)\frac{d}{dx}(g(x)) + g(x)h(x)\frac{d}{dx}(f(x)).$$

In fact, this rule can be extended to any finite number of derivable functions.

(v) *If f and g are differentiable functions at x and h is the function defined by*

$$h(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0, \text{ then } h'(x) = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{(g(x))^2}$$

$$\text{i.e. } \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).\frac{d}{dx}(f(x)) - f(x).\frac{d}{dx}(g(x))}{(g(x))^2},\ g(x) \neq 0.$$

*Thus, the derivative of the quotient of two differentiable functions*

$$= \frac{\textit{deno.} \times \textit{derivative of num.} - \textit{num.} \times \textit{derivative of deno.}}{(\textit{denominator})^2}.$$

This is known as **quotient rule.**

**6.** *If f is a differentiable function at x, then* $\frac{d}{dx}((f(x))^n) = n(f(x))^{n-1}.f'(x)$, *where n is any positive integer.*

**7. Derivative of polynomials**

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$ be a polynomial function, where $a_0, a_1, a_2, \ldots, a_n$ are fixed real numbers and $n$ is a positive integer. Then the derivative of the polynomial $f(x)$ is given by

$$f'(x) = na_0x^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \ldots + a_{n-1}.$$

[Proof of this result is just the application of the above results 1, 2, 3 and 5(*i*), (*ii*).]

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Find the derivatives of the following functions:*

(*i*) $3x^7 - 5x^2 + 9$ (*ii*) $\frac{x}{5} + \frac{5}{x}$ (*iii*) $(ax^2 + b)^2$.

**Solution.** (*i*) Let $f(x) = 3x^7 - 5x^2 + 9$, diff. w.r.t. $x$, we get

$$f'(x) = \frac{d}{dx}(3x^7 - 5x^2 + 9) = \frac{d}{dx}(3x^7) + \frac{d}{dx}(-5x^2) + \frac{d}{dx}(9)$$

$$= 3.\frac{d}{dx}(x^7) + (-5).\frac{d}{dx}(x^2) + 0 = 3.7x^6 - 5.2x^1$$

$$= 21x^6 - 10x.$$

(*ii*) Let $f(x) = \frac{x}{5} + \frac{5}{x}$, diff. w.r.t. $x$, we get

$$f'(x) = \frac{d}{dx}\left(\frac{x}{5} + \frac{5}{x}\right) = \frac{d}{dx}\left(\frac{x}{5}\right) + \frac{d}{dx}\left(\frac{5}{x}\right)$$

$$= \frac{1}{5}\cdot\frac{d}{dx}(x) + 5.\frac{d}{dx}(x^{-1}) = \frac{1}{5}.1 + 5.(-1).x^{-2}$$

$$= \frac{1}{5} - \frac{5}{x^2}, x \neq 0.$$

(*iii*) Let $f(x) = (ax^2 + b)^2 = a^2x^4 + 2abx^2 + b^2$, diff. w.r.t. $x$, we get

$$f'(x) = \frac{d}{dx}(a^2x^4) + \frac{d}{dx}(2abx^2) + \frac{d}{dx}(b^2)$$

$$= a^2.\frac{d}{dx}(x^4) + 2ab.\frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$$

$$= a^2.4x^3 + 2ab.2x^1 + 0 = 4a^2x^3 + 4abx$$

$$= 4ax(ax^2 + b).$$

**Example 2.** *Find the derivative of* $x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n$, *where a is some fixed real number and n is any positive integer.*

**Solution.** Let $f(x) = x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n$, diff. w.r.t. $x$, we get

$$f'(x) = \frac{d}{dx}(x^n + ax^{n-1} + a^2x^{n-2} + \ldots + a^{n-1}x + a^n)$$

$$= \frac{d}{dx}(x^n) + a\frac{d}{dx}(x^{n-1}) + a^2\frac{d}{dx}(x^{n-2}) + \ldots + a^{n-1}\frac{d}{dx}(x) + \frac{d}{dx}(a^n)$$

$$= nx^{n-1} + a.(n-1)x^{n-2} + a^2.(n-2)x^{n-3} + \ldots + a^{n-1}.1 + 0$$

$$= nx^{n-1} + (n-1)ax^{n-2} + (n-2)a^2x^{n-3} + \ldots + a^{n-1}.$$

**Example 3.** *Find the derivatives of the following functions :*

(*i*) $x^{-4}(3 - 4x^{-5})$ (*ii*) $\left(x + \frac{1}{x}\right)^3$

**Solution.** (*i*) Let $f(x) = x^{-4}(3 - 4x^{-5}) = 3x^{-4} - 4x^{-9}$, diff. w.r.t. $x$, we get

$$f'(x) = 3 \times (-4)x^{-5} - 4 \times (-9)x^{-10}$$

$$= -12x^{-5} + 36x^{-10} = -12x^{-5}(1 - 3x^{-5}).$$

(*ii*) Let $f(x) = \left(x + \frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3x \times \frac{1}{x}\left(x + \frac{1}{x}\right)$

$$= x^3 + x^{-3} + 3x + 3x^{-1}, \text{ diff. w.r.t. } x, \text{ we get}$$

$$f'(x) = 3x^2 + (-3)x^{-4} + 3 \times 1 + 3(-1)x^{-2}$$

$$= 3x^2 - \frac{3}{x^4} + 3 - \frac{3}{x^2}.$$

**Example 4.** *If* $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$*, prove that* $2x\dfrac{dy}{dx} + y = 2\sqrt{x}$.

**Solution.** Given $y = \sqrt{x} + \dfrac{1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$, diff. w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} + \left(-\frac{1}{2}\right)x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$$

$$\Rightarrow \quad 2x\frac{dy}{dx} = \sqrt{x} - \frac{1}{\sqrt{x}} \Rightarrow 2x\frac{dy}{dx} + y = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) + \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$$

$$\Rightarrow \quad 2x\frac{dy}{dx} + y = 2\sqrt{x}.$$

**Example 5.** *If* $y = \sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{a}{x}}$*, prove that* $2xy\dfrac{dy}{dx} = \dfrac{x}{a} - \dfrac{a}{x}$.

**Solution.** Given $y = \sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{a}{x}} = \dfrac{1}{\sqrt{a}}x^{1/2} + \sqrt{a}x^{-1/2}$, diff. w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{a}}\cdot\frac{1}{2}x^{-1/2} + \sqrt{a}\cdot\left(-\frac{1}{2}\right).x^{-3/2} = \frac{1}{2\sqrt{a}\sqrt{x}} - \frac{\sqrt{a}}{2x^{3/2}}$$

$$\Rightarrow \quad 2x\frac{dy}{dx} = \frac{\sqrt{x}}{\sqrt{a}} - \frac{\sqrt{a}}{\sqrt{x}} = \sqrt{\frac{x}{a}} - \sqrt{\frac{a}{x}}$$

$$\therefore \quad 2xy\frac{dy}{dx} = \left(\sqrt{\frac{x}{a}} - \sqrt{\frac{a}{x}}\right)\left(\sqrt{\frac{x}{a}} + \sqrt{\frac{a}{x}}\right) = \frac{x}{a} - \frac{a}{x}.$$

**Example 6.** *If* $y = \dfrac{1}{\sqrt{x}}\left(1 - \dfrac{1}{x}\right)$*, prove that* $x^{\frac{5}{2}}\left(2\dfrac{dy}{dx} - y\right) + x^2 - 3 = 0$.

**Solution.** Given $y = \dfrac{1}{\sqrt{x}}\left(1 - \dfrac{1}{x}\right) = x^{-\frac{1}{2}} - x^{-\frac{3}{2}}$, diff. w.r.t. $x$, we get

$$\frac{dy}{dx} = -\frac{1}{2}.x^{-\frac{3}{2}} - \left(-\frac{3}{2}\right)x^{-\frac{5}{2}} = -\frac{1}{2}x^{-\frac{3}{2}} + \frac{3}{2}x^{-\frac{5}{2}}$$

$$\Rightarrow \quad 2\frac{dy}{dx} = -x^{-\frac{3}{2}} + 3x^{-\frac{5}{2}}$$

$$\therefore \quad x^{\frac{5}{2}}\left(2\frac{dy}{dx} - y\right) + x^2 - 3 = x^{\frac{5}{2}}\left(-x^{-\frac{3}{2}} + 3x^{-\frac{5}{2}} - \left(x^{-\frac{1}{2}} - x^{-\frac{3}{2}}\right)\right) + x^2 - 3$$

$$= x^{\frac{5}{2}}\left(3x^{-\frac{5}{2}} - x^{-\frac{1}{2}}\right) + x^2 - 3 = 3 - x^2 + x^2 - 3 = 0.$$

**Example 7.** *Find the derivatives of the following functions :*

(*i*) $(x^2 + 1)(x - 2)$ (*ii*) $\dfrac{3x + 4}{5x^2 - 7x + 9}$.

**Solution.** (*i*) Let $f(x) = (x^2 + 1)(x - 2) = x^3 - 2x^2 + x - 2$, diff. w.r.t. $x$, we get
$f'(x) = 3x^2 - 2.\ 2x^1 + 1 - 0 = 3x^2 - 4x + 1.$

***Alternatively***

$f(x) = (x^2 + 1)(x - 2)$, diff. w.r.t. $x$, we get

$$f'(x) = (x^2 + 1) \cdot \frac{d}{dx}(x - 2) + (x - 2) \cdot \frac{d}{dx}(x^2 + 1) \qquad \text{(product rule)}$$

$$= (x^2 + 1)(1 - 0) + (x - 2)(2x + 0)$$

$$= x^2 + 1 + 2x^2 - 4x = 3x^2 - 4x + 1.$$

(*ii*) Let $f(x) = \dfrac{3x + 4}{5x^2 - 7x + 9}$, diff. w.r.t. $x$, we get

$$f'(x) = \frac{(5x^2 - 7x + 9)\frac{d}{dx}(3x + 4) - (3x + 4)\frac{d}{dx}(5x^2 - 7x + 9)}{(5x^2 - 7x + 9)^2} \qquad \text{(quotient rule)}$$

$$= \frac{(5x^2 - 7x + 9)(3 \times 1 + 0) - (3x + 4)(5 \times 2x - 7 \times 1 + 0)}{(5x^2 - 7x + 9)^2}$$

$$= \frac{3(5x^2 - 7x + 9) - (3x + 4)(10x - 7)}{(5x^2 - 7x + 9)^2}$$

$$= \frac{15x^2 - 21x + 27 - (30x^2 + 19x - 28)}{(5x^2 - 7x + 9)^2} = \frac{55 - 40x - 15x^2}{(5x^2 - 7x + 9)^2}.$$

**Example 8.** *Find the derivatives of the following functions :*

(*i*) $\dfrac{x^n - a^n}{x - a}$ (*ii*) $\dfrac{ax + b}{px^2 + qx + r}$.

**Solution.** (*i*) Let $f(x) = \dfrac{x^n - a^n}{x - a}$, diff. w.r.t. $x$, we get

$$f'(x) = \frac{(x - a).\frac{d}{dx}(x^n - a^n) - (x^n - a^n).\frac{d}{dx}(x - a)}{(x - a)^2} \qquad \text{(quotient rule)}$$

$$= \frac{(x - a)(nx^{n-1} - 0) - (x^n - a^n)(1 - 0)}{(x - a)^2}$$

$$= \frac{nx^n - nax^{n-1} - x^n + a^n}{(x - a)^2} = \frac{(n - 1)x^n - nax^{n-1} + a^n}{(x - a)^2}, \; x \neq a.$$

(*ii*) Let $f(x) = \dfrac{ax + b}{px^2 + qx + r}$, diff. w.r.t. $x$, we get

$$f'(x) = \frac{(px^2 + qx + r).\frac{d}{dx}(ax + b) - (ax + b).\frac{d}{dx}(px^2 + qx + r)}{(px^2 + qx + r)^2} \qquad \text{(quotient rule)}$$

$$= \frac{(px^2 + qx + r)(a.1 + 0) - (ax + b)(p.2x + q.1 + 0)}{(px^2 + qx + r)^2}$$

$$= \frac{a(px^2 + qx + r) - (ax + b)(2px + q)}{(px^2 + qx + r)^2} = -\frac{apx^2 + 2bpx + (bq - ar)}{(px^2 + qx + r)^2}.$$

**Example 9.** *Find the derivatives of the following functions :*

(*i*) $(ax + b)(cx + d)^2$ (*ii*) $(ax + b)^n (cx + d)^m$, $n, m \in \mathbf{N}$.

**Solution.** (*i*) Let $f(x) = (ax + b)(cx + d)^2$, diff. w.r.t. $x$, we get

$$f'(x) = (ax + b) \cdot \frac{d}{dx}((cx + d)^2) + (cx + d)^2 \cdot \frac{d}{dx}(ax + b) \quad \text{(product rule)}$$

$$= (ax + b) \cdot 2(cx + d)^1 \cdot c + (cx + d)^2 \cdot (a \cdot 1 + 0)$$

$$\left(\because \frac{d}{dx}((ax + b)^n) = n(ax + b)^{n-1} \cdot a, \text{ result 4}\right)$$

$$= 2c(ax + b)(cx + d) + a(cx + d)^2$$

$$= (2c(ax + b) + a(cx + d))(cx + d)$$

$$= (3acx + 2bc + ad)(cx + d).$$

(*ii*) Let $f(x) = (ax + b)^n (cx + d)^m$, $n, m \in \mathbf{N}$.

Differentiating w.r.t. $x$, we get

$$f'(x) = (ax + b)^n \cdot \frac{d}{dx}((cx + d)^m) + (cx + d)^m \cdot \frac{d}{dx}((ax + b)^n) \quad \text{(product rule)}$$

$$= (ax + b)^n \cdot m(cx + d)^{m-1} \cdot c + (cx + d)^m \cdot n(ax + b)^{n-1} \cdot a \quad \text{(using result 4)}$$

$$= (ax + b)^{n-1}(cx + d)^{m-1}(mc(ax + b) + na(cx + d))$$

$$= (ax + b)^{n-1}(cx + d)^{m-1}(ac(m + n)x + (mbc + nad)).$$

## EXERCISE 11.2

*Find the derivatives of the following (1 to 5) functions w.r.t. x :*

**1.** (*i*) $\frac{x}{2} + \frac{2}{x}$ (*ii*) $4\sqrt{x} - 2$ (*iii*) $ax^3 + bx^2 + cx + d$ (*iv*) $2x^4 + x$

**2.** (*i*) $\left(2x + \frac{1}{x}\right)^2$ (*ii*) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$.

**3.** (*i*) $x^{-3}(5 + 3x)$ (*ii*) $x^5(3 - 6x^{-9})$.

**4.** (*i*) $(2x + 3)(5x^2 - 7x + 1)$ (*ii*) $(2x - 7)^2(3x + 5)^3$

**5.** (*i*) $\frac{(x + 1)(x - 2)}{\sqrt{x}}$ (*ii*) $\frac{(x + 5)(2x^2 - 1)}{x}$.

**6.** For the function $f$, given by $f(x) = 2x^2 - 3x + 7$, show that $f'(2) = 5f'(1)$.

**7.** If for the function of $f$ defined by $f(x) = kx^2 - 5x + 4$, $f'(3) = 37$, find the value of $k$.

**8.** If $y = x + \frac{1}{x}$, prove that $x^2\frac{dy}{dx} - xy + 2 = 0$.

*Find the derivatives of the following (9 to 11) functions :*

**9.** (*i*) $\frac{x - a}{x - b}$ (*ii*) $\frac{ax + b}{cx + d}$. **10.** (*i*) $\frac{(2x + 1)(3x - 1)}{x + 5}$ (*ii*) $\frac{1}{ax^2 + bx + c}$.

**11.** (*i*) $\frac{px^2 + qx + r}{ax + b}$ (*ii*) $\frac{ax^2 + bx + c}{px^2 + qx + r}$.

### Answers

**1.** (*i*) $\frac{1}{2} - \frac{2}{x^2}$ (*ii*) $\frac{2}{\sqrt{x}}$ (*iii*) $3ax^2 + 2bx + c$ (*iv*) $8x^3 + 1$

**2.** (*i*) $8x - \frac{2}{x^3}$ (*ii*) $1 - \frac{1}{x^2}$ **3.** (*i*) $-15x^{-4} - 6x^{-3}$ (*ii*) $15x^4 + 24x^{-5}$

4. (*i*) $30x^2 + 2x - 19$ (*ii*) $(2x - 7)(3x + 5)^2(30x - 43)$

5. (*i*) $\frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{x}} + \frac{1}{x^{3/2}}$ (*ii*) $4x + 10 + \frac{5}{x^2}$ 7. 7

9. (*i*) $\frac{a-b}{(x-b)^2}$ (*ii*) $\frac{ad-bc}{(cx+d)^2}$ 10. (*i*) $\frac{6(x^2+10x+1)}{(x+5)^2}$ (*ii*) $-\frac{2ax+b}{(ax^2+bx+c)^2}$

11. (*i*) $\frac{apx^2 + 2bpx + bq - ar}{(ax+b)^2}$ (*ii*) $\frac{(aq-bp)x^2 + 2(ar-pc)x + br - cq}{(px^2+qx+r)^2}$

## DERIVATIVES OF COMPOSITE FUNCTIONS

**Theorem.** *If $u = g(x)$ is differentiable at x and $y = f(u)$ is differentiable at u, then y is differentiable at x and*

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$ (We accept it without proof)

**Another form.** The above theorem can also be stated as:

*If g is differentiable at x and f is differentiable at g(x), then the composite function $h(x) = f(g(x))$ is differentiable at x and $h'(x) = f'(g(x)).g'(x)$.*

**Chain Rule.** The above rule is called the **chain rule** of differentiation, since determining the derivative of $y = f(g(x))$ at $x$ involves the following chain of steps:

(*i*) *First, find the derivative of the outer function f at g(x).*

(*ii*) *Second, find the derivative of the inner function g at x.*

(*iii*) *The product of these two derivatives gives the required derivative of the composite function fog at x.*

The chain rule is a very important and useful tool in differentiation. The reader is advised to have sufficient practice in the use of this rule.

**The chain rule works perfectly well to more than two functions.**

**Corollary 1.** *If* $h(x) = (f(x))^n$ *and f is differentiable at x, then*

$$h'(x) = n(f(x))^{n-1}.\, f'(x).$$

**Proof.** Let $f(x) = u$ and $g(u) = u^n$, then

$h(x) = u^n = g(u) = g(f(x))$, differentiate using chain rule

$\therefore$ $h'(x) = g'(f(x))\, f'(x)$ (since $u = f(x)$ and $g'(u)$ exists)

$= g'(u).\, f'(x) = nu^{n-1} f'(x)$

$= n(f(x))^{n-1} f'(x).$

**Corollary 2.** $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, *provided* $\frac{dx}{dt} \neq 0$.

This result is very useful in dealing with **parametric functions.**

**Corollary 3.** $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$, *provided* $\frac{dx}{dt} \neq 0$.

This result is very useful in dealing with **inverse functions.**

**Corollary 4.** $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$

### Derivative of absolute value function

Let $y = |x| = \sqrt{x^2}$, differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{2}(x^2)^{-\frac{1}{2}}.\frac{d}{dx}(x^2) = \frac{1}{2\sqrt{x^2}}.2x = \frac{x}{|x|},\ x \neq 0.$$

*Thus,* $\frac{d}{dx}(|x|) = \frac{x}{|x|}$, $x \neq 0$.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Differentiate the following functions w.r.t. $x$ :*

(i) $\sqrt{2x-3}, x > \frac{3}{2}$ (ii) $(3x^2 - 5x + 1)^7$.

**Solution.** (i) Let $f(x) = \sqrt{2x-3} = (2x-3)^{1/2}$, differentiating w.r.t. $x$, we get

$$f'(x) = \frac{1}{2}(2x-3)^{-1/2}.\frac{d}{dx}(2x-3)$$

$$= \frac{1}{2}(2x-3)^{-1/2}(2.1-0) = \frac{1}{\sqrt{2x-3}},\ x > \frac{3}{2}.$$

(ii) Let $y = (3x^2 - 5x + 1)^7$, differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = 7.(3x^2-5x+1)^6.\frac{d}{dx}(3x^2-5x+1)$$

$$= 7(3x^2-5x+1)^6(3.2x^1 - 5.1 + 0)$$

$$= 7(3x^2-5x+1)^6(6x-5).$$

**Example 2.** *Differentiate the following function w.r.t. $x$:* $\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x}$.

**Solution.** Let $y = \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x} = \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x} \times \frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}-x}$

$$= \frac{(x^2+1)+x^2-2x\sqrt{x^2+1}}{(x^2+1)-x^2} = \frac{2x^2+1-2x\sqrt{x^2+1}}{1}$$

$= 2x^2 + 1 - 2x\sqrt{x^2+1}$, differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = 2.2x + 0 - 2\left[x.\frac{1}{2}(x^2+1)^{-1/2}.2x + \sqrt{x^2+1}.1\right]$$

$$= 4x - 2\left[\frac{x^2}{\sqrt{x^2+1}} + \sqrt{x^2+1}\right] = 4x - 2.\frac{x^2+(x^2+1)}{\sqrt{x^2+1}} = 2\left[2x - \frac{2x^2+1}{\sqrt{x^2+1}}\right].$$

**Example 3.** *If $y = (x + \sqrt{x^2+a^2})^n$, prove that $\frac{dy}{dx} = \frac{ny}{\sqrt{x^2+a^2}}$.*

**Solution.** Given $y = (x + \sqrt{x^2+a^2})^n$ ...(i)

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = n(x+\sqrt{x^2+a^2})^{n-1}.\frac{d}{dx}(x+\sqrt{x^2+a^2})$$

$$= n(x+\sqrt{x^2+a^2})^{n-1}.\left[1+\frac{1}{2}(x^2+a^2)^{-1/2}.2x\right]$$

$$= n\,(x + \sqrt{x^2 + a^2})^{n-1}\left(1 + \frac{x}{\sqrt{x^2 + a^2}}\right) = n\,(x + \sqrt{x^2 + a^2})^{n-1}\,\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}$$

$$= n.\frac{(x + \sqrt{x^2 + a^2})^n}{\sqrt{x^2 + a^2}} = \frac{ny}{\sqrt{x^2 + a^2}} \qquad \text{(using } (i)\text{)}$$

**Example 4.** *If* $y = (x + \sqrt{x^2 - 1})^m$, *prove that* $(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = m^2y^2$.

**Solution.** Given $y = (x + \sqrt{x^2 - 1})^m$ ...$(i)$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = m.(x + \sqrt{x^2 - 1})^{m-1}.\left[1 + \frac{1}{2}(x^2 - 1)^{-1/2}.2x\right]$$

$$= m.(x + \sqrt{x^2 - 1})^{m-1}.\left(1 + \frac{x}{\sqrt{x^2 - 1}}\right) = m.(x + \sqrt{x^2 - 1})^{m-1}.\frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}}$$

$$= m.\frac{(x + \sqrt{x^2 - 1})^m}{\sqrt{x^2 - 1}} = \frac{my}{\sqrt{x^2 - 1}} \qquad \text{(using } (i)\text{)}$$

On squaring both sides, we get

$$\left(\frac{dy}{dx}\right)^2 = \frac{m^2y^2}{x^2 - 1} \Rightarrow (x^2 - 1)\left(\frac{dy}{dx}\right)^2 = m^2y^2.$$

**Example 5.** *Differentiate* $|x^2 - 5|$ *w.r.t.* $x$.

**Solution.** Let $y = |x^2 - 5|$, differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{x^2 - 5}{|x^2 - 5|}.\frac{d}{dx}(x^2 - 5) \qquad \left(\because \frac{d}{dx}(|x|) = \frac{x}{|x|}, x \neq 0\right)$$

$$= \frac{x^2 - 5}{|x^2 - 5|}.(2x - 0) = \frac{2x\,(x^2 - 5)}{|x^2 - 5|},\ x \neq \pm\sqrt{5}.$$

## EXERCISE 11.3

1. (*i*) If $f(x) = \sqrt{4 - x^2}$, $x \in (-2, 2)$, find $f'(x)$.

   (*ii*) If $f(x) = \sqrt{1 - x^2}$, $x \in (0, 1)$, find $f'(x)$.

*Differentiate the following (2 to 4) w.r.t. x :*

2. (*i*) $\sqrt{2x - 1}$ (*ii*) $(3x^2 - 9x + 5)^9$.

3. (*i*) $\sqrt{3x + 2} + \frac{1}{\sqrt{2x^2 + 4}}$ (*ii*) $\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$.

4. (*i*) $\frac{\sqrt{a + x} - \sqrt{a - x}}{\sqrt{a + x} + \sqrt{a - x}}$ (*ii*) $\frac{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}{\sqrt{x^2 + 1} - \sqrt{x^2 - 1}}$.

5. Find the derivative of $|2x^2 - 3|$ with respect to $x$.

### Answers

1. (*i*) $-\frac{x}{\sqrt{4 - x^2}}$ (*ii*) $-\frac{x}{\sqrt{1 - x^2}}$

2. (*i*) $\frac{1}{\sqrt{2x - 1}}, x > \frac{1}{2}$ (*ii*) $27(2x - 3)(3x^2 - 9x + 5)^8$

3. (i) $\frac{3}{2\sqrt{3x+2}} - \frac{2x}{(2x^2+4)^{3/2}}, x > -\frac{2}{3}$ (ii) $-\frac{2a^2x}{\sqrt{a^2-x^2}(a^2+x^2)^{3/2}}$

4. (i) $\frac{a(a-\sqrt{a^2-x^2})}{x^2\sqrt{a^2-x^2}}$ (ii) $2x + \frac{2x^3}{\sqrt{x^4-1}}$ 5. $\frac{4x(2x^2-3)}{|2x^2-3|}$

## IMPLICIT DIFFERENTIATION

If $y$ be a function of $x$ defined by an equation such as

$$y = 7x^4 - 5x^3 + 11x^2 + \sqrt{2}\,x - 3 \quad \ldots(i)$$

$y$ is said to be defined **explicitly** in terms of $x$ and we write $y = f(x)$ where

$$f(x) = 7x^4 - 5x^3 + 11x^2 + \sqrt{2}\,x - 3.$$

However, if $x$ and $y$ are connected by an equation of the form

$$x^4y^3 - 3x^3y^5 + 7y^3 - 8x^2 + 9 = 0 \quad \ldots(ii)$$

*i.e.* $f(x, y) = 0$, then $y$ cannot be expressed explicitly in terms of $x$. But, still the value of $y$ depends upon that of $x$ and there may exist one or more functions '$f$' connecting $y$ with $x$ so as to satisfy equation (*ii*) or there may not exist any of the functions satisfying equation (*ii*).

***For example*** : Consider the equations

$$x^2 + y^2 - 25 = 0 \quad \ldots(iii)$$

and $$x^2 + y^2 + 25 = 0 \quad \ldots(iv)$$

In equation (*iii*), $y$ may be expressed explicitly in terms of $x$, but $y$ is not a function of $x$. Here, we have two functions of $x$ (or two functions of $y$ if $y$ were considered to be independent variable) $f_1$ and $f_2$ defined by $f_1(x) = \sqrt{25 - x^2}$ and $f_2(x) = -\sqrt{25 - x^2}$ which satisfy equation (*iii*).

In equation (*iv*), there are no real values of $x$ that can satisfy it.

In cases (*ii*), (*iii*) and (*iv*), we say that $y$ is an **implicit function** of $x$ (or $x$ is an *implicit function* of $y$) and in all such cases, we find the derivative of $y$ with regard to $x$ (or the derivative of $x$ with regard to $y$) by the process called **implicit differentiation.** Of course, wherever we *differentiate implicitly* an equation that defines one variable as an implicit function of another variable, we shall assume that the function is differentiable.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Find* $\frac{dy}{dx}$ *when* $x^2 + xy + y^2 = 100.$

**Solution.** Given $x^2 + xy + y^2 = 100$.

Regarding $y$ as a function of $x$, differentiating both sides w.r.t. $x$, we get

$$2x + \left(x.\frac{dy}{dx} + y.1\right) + 2y\frac{dy}{dx} = 0 \Rightarrow (x + 2y)\frac{dy}{dx} = -2x - y$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x + y}{x + 2y}.$$

**Example 2.** *If* $x^{2/3} + y^{2/3} = a^{2/3}$, *find* $\frac{dy}{dx}$.

**Solution.** Given $x^{2/3} + y^{2/3} = a^{2/3} \quad \ldots(i)$

Differentiating both sides of (*i*) w.r.t. $x$, regarding $y$ as a function of $x$, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}.\frac{dy}{dx} = 0 \Rightarrow \frac{1}{x^{1/3}} + \frac{1}{y^{1/3}}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = -\sqrt[3]{\frac{y}{x}}.$$

## EXERCISE 11.4

*Find* $\frac{dy}{dx}$ *in the following (1 to 4) :*

**1.** (*i*) $x - y = \pi$ (*ii*) $x^2 + y^2 = 25.$ **2.** (*i*) $xy = c^2$ (*ii*) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2020.$

**3.** (*i*) $\sqrt{x} + \sqrt{y} = \sqrt{c}$ (*ii*) $x^3 + x^2y + xy^2 + y^3 = 81.$

**4.** (*i*) $(x^2 + y^2)^2 = xy$ (*ii*) $\frac{x^2 - y^2}{x^2 + y^2} = c$

**5.** If $x^{2/3} + y^{2/3} = 2$, find $\frac{dy}{dx}$ at (1, 1).

### Answers

**1.** (*i*) 1 (*ii*) $-\frac{x}{y}$ **2.** (*i*) $-\frac{y}{x}$ (*ii*) $-\frac{b^2x}{a^2y}$

**3.** (*i*) $-\sqrt{\frac{y}{x}}$ (*ii*) $-\frac{3x^2 + 2xy + y^2}{x^2 + 2xy + 3y^2}$

**4.** (*i*) $\frac{y - 4x^3 - 4xy^2}{4x^2y + 4y^3 - x}$ (*ii*) $\frac{y}{x}$ **5.** $-1$

### Derivatives of exponential and logarithmic functions

***(i) Derivative of $e^x$ is $e^x$, for all $x \in R$.***

**Proof.** Let $f(x) = e^x$, for all $x \in \mathbf{R}$.

By def., $f'(x) = \underset{h \to 0}{\text{Lt}} \frac{f(x+h) - f(x)}{h} = \underset{h \to 0}{\text{Lt}} \frac{e^{x+h} - e^x}{h}$

$$= \underset{h \to 0}{\text{Lt}} \frac{e^x(e^h - 1)}{h} = e^x . \underset{h \to 0}{\text{Lt}} \frac{e^h - 1}{h}$$

$$= e^x . 1 \qquad \left(\because \underset{h \to 0}{\text{Lt}} \frac{e^h - 1}{h} = 1\right)$$

$$= e^x.$$

***Thus, $\frac{d}{dx}(e^x) = e^x$, for all $x \in R$.***

***(ii) Derivative of $\log x$ is $\frac{1}{x}$, where $x > 0$.***

**Proof.** Let $f(x) = \log x$, $x > 0$.

By def., $f'(x) = \underset{h \to 0}{\text{Lt}} \frac{\log(x+h) - \log x}{h} = \underset{h \to 0}{\text{Lt}} \frac{\log \frac{x+h}{x}}{h}$

$$= \underset{h \to 0}{\text{Lt}} \frac{\log\left(1 + \frac{h}{x}\right)}{h} = \underset{h \to 0}{\text{Lt}} \frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x} . x}$$

$$= \frac{1}{x} \underset{\frac{h}{x} \to 0}{\text{Lt}} \frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \qquad \left(\because h \to 0 \Rightarrow \frac{h}{x} \to 0\right)$$

$$= \frac{1}{x} \cdot 1 \qquad \left(\because \operatorname*{Lt}_{x\to 0} \frac{\log(1+x)}{x} = 1\right)$$

$$= \frac{1}{x}.$$

*Thus,* $\frac{d}{dx}(\log x) = \frac{1}{x}$, *where* $x > 0$.

## Some important deductions

**1. Derivative of $a^x$, $a > 0$, $a \neq 1$.**

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \log a}) \qquad (\because a^x = e^{x \log a})$$

$$= e^{x \log a} \cdot \frac{d}{dx}(x \log a) = e^{x \log a} \cdot \log a \cdot \frac{d}{dx}(x)$$

$$= a^x \log a \cdot 1 = a^x \log a.$$

*Thus,* $\frac{d}{dx}(a^x) = a^x \log a$, $a > 0$, $a \neq 1$.

**2. Derivative of $\log_a x$, $x > 0$, $a > 0$, $a \neq 1$.**

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\log x}{\log a}\right) \qquad \text{(Base changing formula)}$$

$$= \frac{1}{\log a} \cdot \frac{d}{dx}(\log x) = \frac{1}{\log a} \cdot \frac{1}{x} = \frac{1}{x \log a}.$$

*Thus,* $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$, $x > 0$, $a > 0$, $a \neq 1$.

**3. Derivative of $\log|x|$, $x \neq 0$.**

$$\frac{d}{dx}(\log|x|) = \frac{1}{|x|} \cdot \frac{d}{dx}(|x|) = \frac{1}{|x|} \cdot \frac{x}{|x|}, x \neq 0$$

$$= \frac{x}{|x|^2} = \frac{x}{x^2} = \frac{1}{x}, x \neq 0.$$

*Thus,* $\frac{d}{dx}(\log|x|) = \frac{1}{x}$, $x \neq 0$.

**4. Derivative of $\log_a|x|$, $x \neq 0$, $a > 0$, $a \neq 1$.**

$$\frac{d}{dx}(\log_a|x|) = \frac{d}{dx}\left(\frac{\log|x|}{\log a}\right) \qquad \text{(Base changing formula)}$$

$$= \frac{1}{\log a} \cdot \frac{d}{dx}(\log|x|) = \frac{1}{\log a} \cdot \frac{1}{x} = \frac{1}{x \log a}$$

*Thus,* $\frac{d}{dx}(\log_a|x|) = \frac{1}{x \log a}$, $x \neq 0$, $a > 0$, $a \neq 1$.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Differentiate the following functions:*

(*i*) $5^x + \log x$ (*ii*) $\frac{8^x}{x^8}$.

**Solution.** (*i*) Let $y = 5^x + \log x$, differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = 5^x \log 5 + \frac{1}{x} = 5^x \log 5 + \frac{1}{x}.$$

(ii) Let $y = \frac{8^x}{x^8}$, differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{x^8.8^x.\log 8 - 8^x.8x^7}{(x^8)^2} = \frac{x^7 8^x(x\log 8 - 8)}{x^{16}}$$

$$= \frac{8^x(x\log 8 - 8)}{x^9} = \frac{8^x}{x^8}\left(\log 8 - \frac{8}{x}\right).$$

**Example 2.** *Find the derivatives of the following functions:*

(i) $\log\,(x + \sqrt{x^2 - a^2})$ (ii) $\log\left(\frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}}\right)$.

**Solution.** (i) Let $y = \log\,(x + \sqrt{x^2 - a^2})$, differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 - a^2}}.\frac{d}{dx}(x + \sqrt{x^2 - a^2})$$

$$= \frac{1}{x + \sqrt{x^2 - a^2}}.\left[1 + \frac{1}{2}(x^2 - a^2)^{-1/2}.2x\right] = \frac{1}{x + \sqrt{x^2 - a^2}}.\left(1 + \frac{x}{\sqrt{x^2 - a^2}}\right)$$

$$= \frac{1}{x + \sqrt{x^2 - a^2}}.\frac{\sqrt{x^2 - a^2} + x}{\sqrt{x^2 - a^2}} = \frac{1}{\sqrt{x^2 - a^2}}.$$

(ii) Let $y = \log\left(\frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}}\right) = \log\left(\frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} \times \frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}}\right)$

$$= \log\left(\frac{(x+1) + (x-1) + 2\sqrt{x+1}\sqrt{x-1}}{(x+1) - (x-1)}\right) = \log\left(\frac{2x + 2\sqrt{x^2 - 1}}{2}\right)$$

$= \log\,(x + \sqrt{x^2 - 1})$, differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 - 1}}.\left[1 + \frac{1}{2}(x^2 - 1)^{-1/2}.2x\right]$$

$$= \frac{1}{x + \sqrt{x^2 - 1}}.\left(1 + \frac{x}{\sqrt{x^2 - 1}}\right) = \frac{1}{x + \sqrt{x^2 - 1}}.\frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}}.$$

**Example 3.** *If* $y = e^{3\log x + 2x}$, *prove that* $\frac{dy}{dx} = x^2 e^{2x}\,(2x + 3)$.

**Solution.** Given $y = e^{3\log x + 2x} = e^{\log x^3}\,.\,e^{2x} = x^3\,.\,e^{2x}$ $(\because e^{\log x} = x,\ x > 0)$

Differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = x^3\,.\,e^{2x}\,.\,2 + e^{2x}\,.\,3x^2 = x^2 e^{2x}\,(2x + 3).$$

**Example 4.** *If* $y = 9^{\log_3 x}$, *show that* $\frac{dy}{dx} = 2x.$

**Solution.** Given $y = 9^{\log_3 x} = (3^2)^{\log_3 x} = 3^{2\log_3 x} = 3^{\log_3 x^2}$

$\Rightarrow\quad y = x^2$ $(\because a^{\log_a x} = x)$

$\therefore\quad \frac{dy}{dx} = 2x.$

**Example 5.** (i) *If* $e^x + e^y = e^{x+y}$, *prove that* $\frac{dy}{dx} = -\frac{e^x(e^y - 1)}{e^y(e^x - 1)}$.

(ii) *If* $e^x + e^y = e^{x+y}$, *prove that* $\frac{dy}{dx} = -e^{y-x}$.

**Solution.** (*i*) Given $e^x + e^y = e^{x+y}$ ...(*i*)

Differentiating both sides of (*i*) w.r.t. $x$, regarding $y$ as a function of $x$, we get

$$e^x.1 + e^y.\frac{dy}{dx} = e^{x+y}.\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \quad e^x + e^y\frac{dy}{dx} = e^{x+y} + e^{x+y}\frac{dy}{dx} \Rightarrow (e^y - e^{x+y})\frac{dy}{dx} = e^{x+y} - e^x$$

$$\Rightarrow \quad -e^y(e^x - 1)\frac{dy}{dx} = e^x(e^y - 1) \Rightarrow \frac{dy}{dx} = -\frac{e^x(e^y-1)}{e^y(e^x-1)}.$$

(*ii*) Given $e^x + e^y = e^{x+y}$, dividing both sides by $e^{x+y}$, we get

$e^{-y} + e^{-x} = 1$, diff. w.r.t. $x$

$$e^{-y}.\left(-\frac{dy}{dx}\right) + e^{-x}(-1) = 0 \Rightarrow e^{-y}\frac{dy}{dx} = -e^{-x}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{e^{-x}}{e^{-y}} \Rightarrow \frac{dy}{dx} = -e^{y-x}.$$

**Example 6.** *If* $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$, *prove that* $\dfrac{dy}{dx} = 1 - y^2$.

**Solution.** Given $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ ...(*i*)

$$\therefore \quad \frac{dy}{dx} = \frac{(e^x - e^{-x})(e^x + e^{-x}(-1)) - (e^x + e^{-x})(e^x - e^{-x}(-1))}{(e^x - e^{-x})^2}$$

$$= \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2} = 1 - \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2$$

$= 1 - y^2$ (using (*i*))

**Example 7.** *If* $y = \log\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)$, *prove that* $\dfrac{dy}{dx} = \dfrac{x-1}{2x(x+1)}$.

**Solution.** Given $y = \log\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right) = \log\left(\dfrac{x+1}{\sqrt{x}}\right) = \log(x+1) - \dfrac{1}{2}\log x$,

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x+1}.1 - \frac{1}{2}.\frac{1}{x} = \frac{2x-(x+1)}{2x(x+1)} = \frac{x-1}{2x(x+1)}.$$

**Example 8.** *If* $y\sqrt{x^2+1} = \log(\sqrt{x^2+1} - x)$, *prove that* $(x^2+1)\dfrac{dy}{dx} + xy + 1 = 0$.

**Solution.** Given $y\sqrt{x^2+1} = \log(\sqrt{x^2+1} - x)$ ...(*i*)

Differentiating both sides of (*i*) w.r.t. $x$, regarding $y$ as a function of $x$, we get

$$y.\frac{1}{2\sqrt{x^2+1}}.2x + \sqrt{x^2+1}.\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}-x}.\left[\frac{1}{2\sqrt{x^2+1}}.2x - 1\right]$$

$$\Rightarrow \quad \frac{xy}{\sqrt{x^2+1}} + \sqrt{x^2+1}\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}-x}\left(\frac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}}\right)$$

$$\Rightarrow \quad xy + (x^2+1)\frac{dy}{dx} = -1 \Rightarrow (x^2+1)\frac{dy}{dx} + xy + 1 = 0.$$

**Example 9.** *If* $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots \infty}}}$, *prove that* $x(2y-1)\dfrac{dy}{dx} = 1.$

**Solution.** Given $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots \infty}}}$ ...(*i*)

$\Rightarrow$ $y = \sqrt{\log x + y}$ (using (*i*))

$\Rightarrow$ $y^2 = \log x + y.$

Differentiating w.r.t. $x$, we get

$$2y\frac{dy}{dx} = \frac{1}{x} + \frac{dy}{dx} \Rightarrow (2y-1)\frac{dy}{dx} = \frac{1}{x} \Rightarrow x\,(2y-1)\frac{dy}{dx} = 1.$$

## EXERCISE 11.5

*Differentiate the following (1 to 3) functions w.r.t. x :*

1. (*i*) $10^x + \frac{1}{3}e^x - 2\log x$ (*ii*) $e^{-x}$
2. (*i*) $e^{x^3}$ (*ii*) $\sqrt{e^{\sqrt{x}}}$, $x > 0.$
3. (*i*) $\log(\log x)$, $x > 1$ (*ii*) $\log_7(\log x)$, $x > 1.$
4. (*i*) If $f(1) = 4$ and $f'(1) = 2$, find the value of the derivative of $\log f(e^x)$ w.r.t. $x$ at $x = 0.$

   (*ii*) If $f(x) = e^x g(x)$, $g(0) = 2$ and $g'(0) = 1$, then find $f'(0).$

*Differentiate the following (5 to 6) functions w.r.t. x :*

5. (*i*) $e^x + e^{x^2} + \ldots + e^{x^5}$ (*ii*) $\dfrac{5^x \log x}{x^2+1}$
6. (*i*) $\log\left(\dfrac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}\right)$ (*ii*) $x\sqrt{1+x^2} + \log(x+\sqrt{x^2+1}).$
7. (*i*) If $y = \dfrac{\log(x+\sqrt{x^2+1})}{\sqrt{x^2+1}}$, prove that $(x^2+1)\dfrac{dy}{dx} + xy = 1.$

   (*ii*) If $y = e^{2\log x + 3x}$, prove that $\dfrac{dy}{dx} = x(2+3x)\,e^{3x}.$
8. Find $\dfrac{dy}{dx}$ when

   (*i*) $xy + xe^{-y} + ye^x = x^2$ (*ii*) $e^{x-y} = \log\left(\dfrac{x}{y}\right).$
9. If $y \log x = x - y$, prove that $\dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}.$

   **Hint.** $y\log x = x - y \Rightarrow y\,(1+\log x) = x \Rightarrow y = \dfrac{x}{1+\log x}.$

### Answers

1. (*i*) $10^x \log 10 + \frac{1}{3}e^x - \frac{2}{x}$ (*ii*) $-e^{-x}$
2. (*i*) $3x^2e^{x^3}$ (*ii*) $\dfrac{e^{\sqrt{x}}}{4\sqrt{x}\sqrt{e^{\sqrt{x}}}}$
3. (*i*) $\dfrac{1}{x\log x}$ (*ii*) $\dfrac{1}{x\log 7\log x}$
4. (*i*) $\frac{1}{2}$ (*ii*) 3

5. (i) $e^x + 2xe^{x^2} + 3x^2e^{x^3} + 4x^3e^{x^4} + 5x^4e^{x^3}$ (ii) $\dfrac{5^x((x^2+1)(1+x\log 5\log x) - 2x\log x)}{x(x^2+1)^2}$

6. (i) $\dfrac{2}{\sqrt{x^2-a^2}}$ (ii) $2\sqrt{x^2+1}$

8. (i) $\dfrac{2x - y - e^{-y} - ye^x}{x + e^x - xe^{-y}}$ (ii) $\dfrac{y(xe^{x-y}-1)}{x(ye^{x-y}-1)}$.

## MULTIPLE CHOICE QUESTIONS

***Choose the correct answer from the given four options in questions (1 to 4):***

1. If $f(x) = \dfrac{(3x+1)(2\sqrt{x}-1)}{\sqrt{x}}$, then $f'(1)$ is equal to

(a) 5 (b) $-5$ (c) 6 (d) $\dfrac{11}{2}$

2. If $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, then $\dfrac{dy}{dx}$ at $x = 1$ is

(a) 1 (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{\sqrt{2}}$ (d) 0

3. If $y = \dfrac{1+\dfrac{1}{x^2}}{1-\dfrac{1}{x^2}}$, then $\dfrac{dy}{dx}$ is

(a) $-\dfrac{4x}{(x^2-1)^2}$ (b) $-\dfrac{4x}{x^2-1}$ (c) $\dfrac{1-x^2}{4x}$ (d) $\dfrac{4x}{x^2-1}$

4. If $y = \log\left(\dfrac{1-x^2}{1+x^2}\right)$, $|x| < 1$, then $\dfrac{dy}{dx} =$

(a) $\dfrac{4x}{(x^2-1)^2}$ (b) $-\dfrac{4x}{1-x^4}$ (c) $\dfrac{1-x^2}{4x}$ (d) $\dfrac{4x}{x^2-1}$

### Answers

1. (a) 2. (d) 3. (a) 4. (b)

## FILL IN THE BLANKS

***Fill in the blanks in questions (1 to 8):***

1. If $f(x) = \dfrac{x^{100}}{100} + \dfrac{x^{99}}{99} + \ldots + \dfrac{x^2}{2} + x + 1$, then $f'(1) =$ ........................

2. If $f(x) = \dfrac{x-4}{2\sqrt{x}}$, then $f'(1) =$ ........................

3. If $f(x) = x^{100} + x^{99} + x^{98} + \ldots + x + 1$, then $f'(1) =$ ........................

4. If $f(x) = 2x + 1$, then the derivative of $(fof)(x)$ is ........................

5. For the curve $\sqrt{x} + \sqrt{y} = 1$, $\dfrac{dy}{dx}$ at the point $\left(\dfrac{1}{4}, \dfrac{1}{4}\right)$ is........................

6. The derivative of the function $f(x) = 2^{5x^2}$ is ........................

7. If $x > 0$, then the derivative of $\log_{10} x$ is ........................

8. If $x \neq 0$, then the derivative of $\log_2 |x|$ is ........................

## Answers

1. 100
2. $\frac{5}{4}$
3. 5050
4. 4
5. $-1$
6. $2^{5x^2}.10x\log 2$
7. $\frac{1}{x\log 10}$
8. $\frac{1}{x\log 2}$