

APPLIED MATHEMATICS – XII

Chapter 14: LINEAR PROGRAMMING

1. What is Linear Programming?

Linear Programming (L.P.) is a mathematical method used to determine the optimum value (maximum or minimum) of a linear objective function, subject to a set of linear equations or inequalities called constraints.

In simple words:

Linear Programming helps us choose the best possible plan from several alternatives when resources are limited.

2. Why do we use Linear Programming?

- To maximize profit or output.
- To minimize cost, production time, transportation cost, etc.
- To allocate limited resources in the best possible way.
- To select the best combination of products, investments, foods, routes, etc.

3. Three Classical L.P.P. Problems

1. Manufacturing problems – decide how many units of different products should be produced to maximize profit, subject to limits on labour, machines, raw materials, storage, etc.
2. Diet problems – decide the quantities of different foods so that nutritional requirements are satisfied at minimum cost.
3. Transportation problems – decide how much stock should be transported from different sources to different destinations so that total transportation cost is minimized.

4. Important Terms and Definitions

Objective Function

A linear function which is to be maximized or minimized is called the objective function.

General form:

$$Z = a_1x_1 + a_2x_2 + \dots + a_nx_n$$

Examples: total profit, total return, total cost, or production time.

Decision Variables

The unknown quantities whose values have to be decided are called decision variables. For a two-variable problem, they are usually denoted by x and y .

Constraints

The equations or inequalities imposed on the decision variables are called constraints.

Common forms:

Linear condition: $\dots \leq \dots$

Linear condition: $\dots \geq \dots$

Linear condition: $\dots = \dots$

Linear condition: $\dots < \dots$

Linear condition: $\dots > \dots$

Non-negativity Restrictions

In an L.P.P., decision variables are normally non-negative because quantities such as number of products, money invested, food quantities, etc. cannot be negative.

For two variables: $x \geq 0$, $y \geq 0$

Geometrically, $x \geq 0$ and $y \geq 0$ restrict the solution to the first quadrant of the xy-plane.

Feasible Region

The common region satisfying all the constraints of an L.P.P. is called the feasible region.

Feasible Solution

Every point lying in the feasible region is called a feasible solution.

Convex Feasible Region

The feasible region of a linear programming problem is always a convex set. If any two points of the region are joined by a straight line segment, every point of that segment also lies in the region.

Optimal Solution

A feasible solution that gives the maximum or minimum value of the objective function is called an optimal solution.

- An L.P.P. may have no optimal solution.
- It may have one optimal solution.
- It may have more than one optimal solution.

5. Basic Requirements of Linear Programming

- There must be a well-defined objective to achieve — either maximize or minimize.
- Resources must have limited availability, producing constraints or restrictions.
- There must be a finite number of decision variables and a finite number of constraints.
- All elements of the L.P.P. must be quantifiable.
- The objective function and constraints must be expressible as linear equations or inequalities.
- There must be alternative courses of action from which a plan can be selected.
- Decision variables must satisfy non-negativity restrictions.

6. Advantages of Linear Programming

Linear Programming can help in:

- Choosing the best alternative so that profit is maximized or cost is minimized.

- Allocating manpower, machines, budget, storage and other limited resources efficiently.
- Considering external factors such as market demand and purchasing power.
- Identifying bottlenecks in a production process.
- Choosing suitable production and inventory policies to handle seasonal demand.
- Selecting an effective media mix for advertising.
- Determining suitable travelling routes for salespersons.
- Choosing a crop mix to reduce risk and improve profit.

7. Limitations of Linear Programming

- It generally deals with one objective at a time, whereas real-life decisions may involve several objectives.
- The assumption of proportionality between inputs and outputs may not always hold in practice.
- The linearity assumption may ignore synergies or interactions between activities.
- Many real-life decision variables, such as number of workers, are integral, while an ordinary L.P.P. allows continuous values.
- The model is only as reliable as the data and assumptions used to formulate it.

8. Formulation of an L.P.P. in Two Variables x and y

There are three main steps.

Step 1: Define decision variables

Let x and y represent the quantities of the two activities/products/investments being decided.

Step 2: Form the objective function and constraints

Write the objective as a linear expression:

Maximize or Minimize $Z = ax + by$

Then translate every limitation into a linear equation or inequality.

General form:

$ax + by \leq c$, or $ax + by \geq c$, or $ax + by = c$

Finally include:

$x \geq 0$, $y \geq 0$

Step 3: Find the feasible region and optimum

Represent the constraints graphically, identify the feasible region, find its corner points, and then evaluate the objective function at those points to determine the required maximum or minimum.

9. Example 1 – Manufacturing / Product Mix

A dealer deals in wall hangings and artificial plants. He has ₹15,000 to invest and can store at most 80 pieces. A wall hanging costs ₹300 and gives a profit of ₹50. An artificial plant costs ₹150 and gives a profit of ₹18. Formulate the L.P.P. to maximize profit.

Let x = number of wall hangings and y = number of artificial plants.

Profit:

$$Z = 50x + 18y$$

Investment constraint:

$$300x + 150y \leq 15000$$

Dividing by 150:

$$2x + y \leq 100$$

Storage constraint:

$$x + y \leq 80$$

Non-negativity:

$$x \geq 0, y \geq 0$$

Therefore:

$$\text{Maximize } Z = 50x + 18y$$

subject to $2x + y \leq 100$, $x + y \leq 80$, $x \geq 0$, $y \geq 0$.

10. Example 2 – Investment Problem

A retired person wants to invest up to ₹20,000 in bonds A and B. Bond A gives an 8% return and bond B gives a 7% return. He invests at least ₹5,000 in A, not more than ₹8,000 in B, and at least as much in A as in B. Formulate the L.P.P. to maximize return.

Let x = amount invested in bond A and y = amount invested in bond B.

Return:

$$Z = 0.08x + 0.07y$$

Constraints:

$$x + y \leq 20000$$

$$x \geq 5000$$

$$y \leq 8000$$

$$x \geq y$$

$$x \geq 0, y \geq 0$$

Therefore:

$$\text{Maximize } Z = 0.08x + 0.07y$$

subject to all the above constraints.

11. Example 3 – Manufacturing Problem

A manufacturer produces packages of nuts and bolts. A package of nuts requires 1 hour on machine A and 3 hours on machine B. A package of bolts requires 3 hours on machine A and 1 hour on machine B. Profit is ₹2.50 per package of nuts and ₹1 per package of bolts. Each machine can operate for at most 12 hours. Formulate the L.P.P. to maximize profit.

Let x = number of packages of nuts and y = number of packages of bolts.

Machine A constraint:

$$x + 3y \leq 12$$

Machine B constraint:

$$3x + y \leq 12$$

Profit:

$$Z = 2.50x + y$$

Non-negativity:

$$x \geq 0, y \geq 0$$

Hence:

$$\text{Maximize } Z = 2.50x + y$$

subject to $x + 3y \leq 12$, $3x + y \leq 12$, $x \geq 0$, $y \geq 0$.

12. Quick Method to Formulate Any L.P.P.

1. Identify what has to be decided.
2. Define the decision variables x and y .
3. Write the objective function: profit/return/output $\rightarrow$ maximize; cost/time $\rightarrow$ minimize.
4. Translate each resource limitation into a constraint.
5. Add minimum/maximum requirements such as $x \geq 5000$ or $y \leq 8000$.
6. Always add non-negativity restrictions: $x \geq 0$, $y \geq 0$.
7. Write the final L.P.P. neatly and check every condition.

13. Exam Keywords to Remember

Term	Meaning
Objective function	Linear function to be maximized or minimized
Decision variables	Unknown quantities to be determined
Constraints	Restrictions/limitations on decision variables
Non-negativity	Decision variables are ≥ 0
Feasible region	Common region satisfying all constraints
Feasible solution	Any point in the feasible region

Optimal solution	Feasible solution giving best objective value
L.P.P.	Linear Programming Problem

14. Chapter Summary

Linear Programming is a technique for obtaining the optimum value of a linear objective function under linear constraints. The central ideas are decision variables, objective function, constraints, non-negativity, feasible region and optimal solution. In two-variable L.P.P.s, the constraints can be represented graphically on the xy-plane. Correct formulation is the most important first step: define the variables, form the objective function, translate every condition into a constraint, and include non-negativity restrictions.

Memory Trick: D-O-C-F-O

D → O → C → F → O

Decision variables → Objective function → Constraints → Feasible region → Optimum