

PERMUTATIONS AND COMBINATIONS

1.

Permutation: An arrangement in which order matters.

Example: AB and BA are different arrangements.

Combination: A selection in which order does not matter.

Example: {A, B} and {B, A} represent the same selection.

2. Permutations

Theorem 1: Permutations without repetition

The number of permutations of n different objects taken r at a time is:

$${}^n P_r = n! / (n - r)!, \quad 0 \leq r \leq n$$

Short Proof:

The first position can be filled in n ways, the second in $(n - 1)$ ways, the third in $(n - 2)$ ways, and so on.

Therefore,

$${}^n P_r = n(n - 1)(n - 2) \cdots (n - r + 1) = n! / (n - r)!$$

Hence proved.

Theorem 2: Repetition allowed

The number of permutations of n different objects taken r at a time, when repetition is allowed, is:

$$n^r$$

Short Proof:

Each of the r positions can be filled in n ways. By the Fundamental Principle of Counting,

$$n \times n \times \cdots \times n = n^r$$

Hence proved.

Theorem 3: A particular object is always included

If one particular object must be included in every arrangement of r objects chosen from n different objects, then:

$$r \cdot {}^{n-1}P_{r-1}$$

Short Proof:

The particular object can occupy any one of r positions. The remaining $r - 1$ positions can be filled from the remaining $n - 1$ objects in ${}^{n-1}P_{r-1}$ ways.

$$\text{Required number} = r \cdot {}^{n-1}P_{r-1}$$

Theorem 4: A particular object is never included

$${}^{n-1}P_r$$

Short Proof: Exclude the specified object. We then arrange r objects from the remaining $n - 1$ objects.

Theorem 5: Two specified objects always occur together

$$2!(r - 1) \cdot {}^{n-2}P_{r-2}$$

Treat the two specified objects as one block. The block can occupy $r - 1$ positions, the remaining objects can be arranged

in ${}^{n-2}P_{r-2}$ ways, and the two objects within the block can be interchanged in $2!$ ways.

$$\text{Required number} = 2!(r-1) \cdot {}^{n-2}P_{r-2}$$

3. Permutations When Objects Are Identical

If n objects contain p alike objects of one kind, q alike objects of another kind, r alike objects of a third kind, and the remaining objects are all different, then:

$$\text{Number of arrangements} = n! / (p! q! r!)$$

Reason: Starting with $n!$ arrangements of distinct objects, divide by $p!$, $q!$, $r!$ because interchanging identical objects does not produce a new arrangement.

Example: LEVEL has 5 letters, with L repeated twice and E repeated twice.

$$5! / (2! 2!) = 30$$

4. Important Properties of Permutations

- ${}^nP_0 = 1$
- ${}^nP_n = n!$
- ${}^nP_r = n! / (n-r)!$
- ${}^nP_r = n(n-1)\cdots(n-r+1)$
- ${}^nP_r = n \cdot {}^{n-1}P_{r-1}$
- ${}^nP_r = r \cdot {}^{n-1}P_{r-1} + (n-r) \cdot {}^{n-1}P_r$

5. Combination

A combination is a selection where order does not matter.

$${}^nC_r = n! / [r!(n-r)!]$$

Relation between permutation and combination:

$${}^n P_r = r! \cdot {}^n C_r$$

$$\text{Therefore, } {}^n C_r = {}^n P_r / r!$$

6. Important Properties of Combinations

- **Symmetry:** ${}^n C_r = {}^n C_{n-r}$
- ${}^n C_0 = {}^n C_n = 1$
- ${}^n C_1 = n$
- ${}^n C_r = (n/r) \cdot {}^{n-1} C_{r-1}$
- **Pascal's identity:** ${}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$

7. Formula

Situation	Formula
Arrange r from n	${}^n P_r = n! / (n-r)!$
Repetition allowed	n^r
All n arranged	$n!$
One particular object always included	$r \cdot {}^{n-1} P_{r-1}$
One particular object excluded	${}^{n-1} P_r$
Two specified objects together	$2!(r-1) \cdot {}^{n-2} P_{r-2}$
Identical objects	$n! / (p!q!r!)$
Select r from n	${}^n C_r = n! / [r!(n-r)!]$
Relation	${}^n P_r = r! {}^n C_r$
Symmetry	${}^n C_r = {}^n C_{n-r}$

8. Practice Problems

1. Find ${}^n P_r$ for $n = 7, r = 3$.

2. Find ${}^9P_4$.

3. Find ${}^{10}C_3$.

4. Find ${}^{12}C_2$.

Repetition

5. How many 4-digit numbers can be formed using the digits 1, 2, 3, 4, 5, if repetition is allowed?

6. How many 3-letter codes can be formed using the 6 letters A, B, C, D, E, F, if repetition is allowed?

Conditions

7. How many permutations of 5 objects taken 3 at a time contain a particular object?

8. How many permutations of 7 objects taken 4 at a time do not contain a particular object?

9. How many permutations of 8 objects taken 4 at a time contain two specified objects together?

Identical Objects

10(a). Find the number of distinct arrangements of the letters of BANANA.

10(b). Find the number of distinct arrangements of the letters of SUCCESS.

10(c). Find the number of distinct arrangements of the letters of MISSISSIPPI.

Mixed Problems

- 11. How many 5-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5 without repetition?**
- 12. How many 4-digit even numbers can be formed using 1, 2, 3, 4, 5, 6 without repetition?**
- 13. In how many ways can 4 students be selected from 10 students?**
- 14. In how many ways can 4 students be selected and arranged in four different seats from 10 students?**
- 15. How many arrangements of the letters of OBEDIENCE are possible if all vowels occur together?**

9. Answer Key

- **1. 210**
- **2. 3024**
- **3. 120**
- **4. 66**
- **5. 625**
- **6. 216**
- **7. 60**
- **8. 360**
- **9. 336**
- **10(a). 60**
- **10(b). 420**
- **10(c). 34650**
- **11. 600**
- **12. 180**

- 13. 210
- 14. 5040
- 15. 960

10. Recall

Permutation → Arrangement → Order matters → P

Combination → Selection → Order does not matter → C

$${}^n P_r = r! \cdot {}^n C_r$$